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A TWO-CARDINAL THEOREM

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ABSTRACT. We prove the following theorem and deal with some related questions: If for all $n<\omega$, T has a model M such that $n^n \le |Q^M|^n \le |P^M| < \aleph_0$ then for all λ, μ such that $|T| \le \mu \le \lambda < \mathrm{Ded}^*(\mu)$ (e.g. $\mu = \aleph_0$, $\lambda = 2^{\aleph_0}$), T has a model of type (λ, μ), i.e. $|Q^M| = \mu$, $|P^M| = \lambda$.

1. **Introduction.** We shall deal with first order theories T; for simplicity we let T be countable, except in §3. It is well known that if T has a model of type $(\beth_\omega, \aleph_0)$ (i.e. a model M of power $\beth_\omega$ with $|Q^M| = \aleph_0$), then for every $\lambda > \aleph_0$ T has a model of type $(\lambda, \aleph_0)$. This is designated by $(\beth_\omega, \aleph_0) \to (\lambda, \aleph_0)$. One may ask the question: For what λ does $(\aleph_\omega, \aleph_0) \to (\lambda, \aleph_0)$? In particular does $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$? It is of course impossible to ask for more since there is a sentence having a model of type (λ, μ) iff $\aleph_0 \le \mu \le \lambda \le 2^\mu$ (or iff $\aleph_0 \le \mu \le \lambda < \mathrm{Ded}^*\mu$).

We give a combinatorial lemma which implies $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$ and seems to be equivalent to it assuming $MA + 2^{\aleph_0} > \aleph_\omega$. This Lemma still remains an open problem. We finally prove a related two-cardinal theorem (Theorem 1), of interest in its own right, which was stated in the abstract.

2. **Notation.**

Definition 1. A *tree* is a partially ordered set $(X, <)$ such that for each node $x \in X$ the set of predecessors of x is well ordered by $<$. A *branch* is a maximal chain. The *height* of a branch is its order type (always an ordinal).

Definition 2. Let μ be a cardinal. $\mathrm{Ded}^*(\mu)$ is the first power λ such that there is no tree with $\le \mu$ nodes and $\ge \lambda$ branches of the same height. (In this definition we may assume that all trees are subtrees of $({}^{<\mu^+}2, <)$, the tree of all $0-1$ sequences of length $<\mu^+$, ordered by continuation.)

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For example, $\mathrm{Ded}^*(\aleph_0) = (2^{\aleph_0})^+$ and, in general, $\mathrm{Ded}^*(\mu) \leq (2^\mu)^+$. See Baumgartner [1] for results about Ded* and Ded (which we shall not even define here); in particular, it is consistent that $\mathrm{Ded}^*(\aleph_1) < (2^{\aleph_1})^+$.

Let Q and P be two unary predicates and Q^M, P^M their interpretations in the model M. We vary from standard notation by letting (λ, μ)-model mean a model M with $|P^M| = \lambda$, $|Q^M| = \mu$.

Our main theorem is thus denoted by $\{(m_i, n_i): i < \omega\} \rightarrow (\lambda, \mu)$ for $\aleph_0 \leq \mu \leq \lambda < \mathrm{Ded}^*(\mu)$, $\aleph_0 > m_i > n_i^i \geq i^i$.

η, ν will denote sequences of zeroes and ones; ${}^\alpha 2$ the set of all $0-1$ sequences of length α; $l(\eta)$ the length of η; $\eta\hat{\ }\nu$ the concatenation of η and ν; and $\eta|\beta$ the initial subsequence of η of length β. Let ${}^{<\alpha}2 = \bigcup_{\beta<\alpha} {}^\beta 2$.

3. A two-cardinal theorem. The standard way of proving two-cardinal theorems $(\lambda_0, \mu_0) \rightarrow (\lambda_1, \mu_1)$ is to find a set of sentences Γ such that

(i) if T has a model of type (λ_0, μ_0) then $T \cup \Gamma$ is consistent;

(ii) if $T \cup \Gamma$ is consistent then T has a model of type (λ_1, μ_1).

Assume w.l.o.g. that T is a theory in a language L, and has Skolem functions. We use this method to prove

Theorem 1. *If for all $n < \omega$ every finite subset of T has a model M such that $n^n \leq |Q^M|^n \leq |P^M| < \aleph_0$, then for all λ, μ such that $|T| \leq \mu \leq \lambda < \mathrm{Ded}^*(\mu)$, T has a model of type (λ, μ).*

Notice that for $\mu = \aleph_0$ the conclusion is that T has a model of type $(2^{\aleph_0}, \aleph_0)$ (when T is countable).

Definition 3. Let $\eta_i, \nu_i \in {}^{<\alpha}2$ for $i = 1, \cdots, n$. $\langle \eta_1, \cdots, \eta_n \rangle$ and $\langle \nu_1, \cdots, \nu_n \rangle$ are *similar over* β if for all $i = 1, \cdots, n$, $l(\eta_i), l(\nu_i) \geq \beta$, $\eta_i|\beta = \nu_i|\beta$, and for all i, j, $1 \leq i < j \leq n$, $\eta_i|\beta \neq \eta_j|\beta$ (and thus $\nu_i|\beta \neq \nu_j|\beta$).

Definition 4. Let D be a set of $0-1$ sequences. Define

$$\Gamma_L(D) = \{P(y_\eta): \eta \in D\} \cup \{y_\eta \neq y_\nu: \eta \neq \nu \in D\}$$
$$\cup \{z_1 = \tau(y_{\eta_1}, \cdots, y_{\eta_n}) \wedge z_2 = \tau(y_{\nu_1}, \cdots, y_{\nu_n}) \wedge Q(z_1)$$
$$\rightarrow z_1 = z_2: \tau \text{ is a term in } L,\ \eta_i, \nu_i \in D \text{ and}$$
$$\langle \eta_1, \cdots, \eta_n\rangle \text{ and } \langle \nu_1, \cdots, \nu_n\rangle \text{ are similar over some } \beta\}.$$

Now, by way of fulfilling part (ii) above it is easy to see

Lemma 1. *If $T \cup \Gamma_L(2^\omega)$ is consistent and $|T| \leq \mu \leq \lambda < \mathrm{Ded}^*(\mu)$, then*

T has a model of type (λ, μ_1), *for some* $\mu_1 \leq \mu$. *In particular, if* $T \cup \Gamma_L(2^\omega)$ *is consistent and* M *is the Skolem closure of* $\{y_\eta : \eta \in 2^\omega\}$, *then* M *is of type* $(2^{\aleph_0}, \aleph_0)$.

Let us turn now to part (i). We shall list some conditions which are sufficient for proving the consistency of $T \cup \Gamma_L(2^\omega)$.

By the compactness theorem, it is enough to show the consistency of $T' \cup \Gamma'_L({}^n2)$ (where the prime on $\Gamma_L(D)$ indicates that in the definition of $\Gamma_L(D)$ τ ranges over a *finite* set of terms of L, say $\{\tau_0, \cdots, \tau_{n_0}\}$, each having $\leq n_0$ variables, and T' is a finite subset of T). This holds because we can replace T by $T_1 = T \cup \{Q(c_i): i < \mu\} \cup \{c_i \neq c_j : i < j < \mu\}$, the c_i-new individual constants. T_1 satisfies the hypothesis of Theorem 1, and in every model M of it $|Q^M| \geq \mu$. So by the lemma this is sufficient. This must be shown for all n, $n_0 < \omega$.

Definition 5. Let M be a model, A a subset of M, $\bar{b}$, $\bar{c} \in M$. Define $\bar{b} \sim \bar{c}$ (mod A) if for all $i \leq n_0$ and for any presentation of τ_i, $\tau_i(\bar{x}, \bar{y})$ (i.e., ordering and identification of the variables of τ_i), we have for all $\bar{a} \in A$

$$\tau_i(\bar{c}, \bar{a}) \in Q^M \vee \tau_i(\bar{b}, \bar{a}) \in Q^M \Rightarrow \tau_i(\bar{c}, \bar{a}) = \tau_i(\bar{b}, \bar{a}).$$

If $\bar{b}$ is a single-element sequence we simply write b.

So clearly if the number of such presentations is n_1 (so n_1 depends on n_0 only), then this equivalence relation has $\leq (|Q^M| + 1)^k$ equivalence classes, where $k = |A|^{n_0} n_1$.

Claim 1. Let D be a set of $0 - 1$ sequences of length n and $n - 1$ such that no two sequences are comparable (i.e. no one is an initial segment of the other). Assume that the assignment $\{y_\eta \rightarrow a_\eta : \eta \in D\}$ satisfies $\Gamma'_L(D)$. Let $\nu \in D$ be of length $n - 1$ and let $d \in P^M - \{a_\eta : \eta \in D\}$ be such that $d \sim a_\nu$ (mod$\{a_\eta : \eta \neq \nu, \eta \in D\}$). Let $a_{\nu\frown\langle 0\rangle} = a_\nu$, $a_{\nu\frown\langle 1\rangle} = d$, and $D' = (D - \{\nu\}) \cup \nu\frown\langle 0\rangle, \nu\frown\langle 1\rangle\}$. Then the assignment $\{y_\eta \rightarrow a_\eta : \eta \in D'\}$ satisfies $\Gamma'_L(D')$.

Proof. Let $\langle u_1, \cdots, u_n\rangle$, $\langle v_1, \cdots, v_n\rangle$ be similar over some β ($\leq n$), u_i, $v_i \in D'$. We must show

$$z_1 = \tau(a_{u_1}, \cdots, a_{u_n}) \wedge z_2 = \tau(a_{v_1}, \cdots, a_{v_n}) \wedge Q(z_1) \rightarrow z_1 = z_2,$$

i.e., $\langle a_{u_1}, \cdots, a_{u_n}\rangle \sim \langle a_{v_1}, \cdots, a_{v_n}\rangle$ (mod $\emptyset$).

If $\beta = n$, we have $u_i = v_i$ and the result is trivial. If $\beta \leq n - 1$, then by the definition of similarity, at most one of the v_i's can be $\nu\frown\langle 0\rangle$ or $\nu\frown\langle 1\rangle$; likewise for the u_i's. If *none* of the u_i's or v_i's are $\nu\frown\langle 0\rangle$ or $\nu\frown\langle 1\rangle$, then the result holds by our hypothesis. Thus without loss of generality we may assume $v_1 \in \{\nu\hat{}\langle 0\rangle, \nu\hat{}\langle 1\rangle\}$. Clearly for $i \neq 1$, $u_i, v_i \notin \{\nu\hat{}\langle 0\rangle, \nu\hat{}\langle 1\rangle\}$. Now $a_{v_1} \sim a_\nu \pmod{\{a_\eta : \eta \neq \nu, \eta \in D\}}$, since either $a_{v_1} = a_\nu$ or $a_{v_1} = d$. Thus $\langle a_{v_1}, a_{v_2}, \cdots, a_{v_n}\rangle \sim \langle a_\nu, a_{v_2}, \cdots, a_{v_n}\rangle \pmod{\emptyset}$.

Case 1. $u_1 \in \{\nu\hat{}\langle 0\rangle, \nu\hat{}\langle 1\rangle\}$. Then $\langle a_{u_1}, a_{u_2}, \cdots, a_{u_n}\rangle \sim \langle a_\nu, a_{u_2}, \cdots, a_{u_n}\rangle \pmod{\emptyset}$. Clearly $\langle \nu, u_2, \cdots, u_n\rangle$ and $\langle \nu, v_2, \cdots, v_n\rangle$ are similar over the above β. And so by the assumption on $\Gamma'_L(D)$, $\langle a_\nu, a_{u_2}, \cdots, a_{u_n}\rangle \sim \langle a_\nu, a_{v_2}, \cdots, a_{v_n}\rangle \pmod{\emptyset}$. Thus we have $\langle a_{u_1}, a_{u_2}, \cdots, a_{u_n}\rangle \sim \langle a_{v_1}, a_{v_2}, \cdots, a_{v_n}\rangle \pmod{\emptyset}$.

Case 2. $u_1 \notin \{\nu\hat{}\langle 0\rangle, \nu\hat{}\langle 1\rangle\}$. Then $\langle \nu, v_2, \cdots, v_n\rangle$, $\langle v_1, \cdots, v_n\rangle$, $\langle u_1, \cdots, u_n\rangle$ are all similar over β, so it follows that

$$\langle a_{u_1}, \cdots, a_{u_n}\rangle \sim \langle a_\nu, a_{v_2}, \cdots, a_{v_n}\rangle \sim \langle a_{v_1}, a_{v_2}, \cdots, a_{v_n}\rangle \pmod{\emptyset}.$$

Q.E.D.

Claim 2. In order to show the consistency of $T' \cup \Gamma'_L({}^n2)$ for all $n < \omega$ it is sufficient to prove:

For all $m < \omega$ there is a model M of T' and a sequence of sets $X_1 \subset X_2 \subset \cdots \subset X_m \subset P^M$ such that for all $i = 1, \cdots, m-1$ and all distinct $a_1, \cdots, a_m, a_{m+1} \in X_i$, there is $a'_{m+1} \in X_{i+1}$, $a'_{m+1} \notin \{a_1, \cdots, a_{m+1}\}$, such that $a'_{m+1} \sim a_{m+1} \pmod{\{a_1, \cdots, a_m\}}$.

Proof. This is a corollary of the previous claim by repeated use of it.

Claim 3. Theorem 1 follows from the following combinatorial assertion:

(*) For all $m, k < \omega$ there is $l = l(k, m) < \omega$ such that for all $r < \omega$: if F is an m-place function on a set A of power $|A| = r^l$ whose range is subsets of A of power $\leq r$, then there is $B \subset A$, $|B| = r^k$, such that for all distinct $a_1, \cdots, a_{m+1} \in B$, $a_{m+1} \notin F(a_1, \cdots, a_m)$.

Proof. We will show that the condition of Claim 2 follows from (*) and the hypothesis of Theorem 1. Let $l(k, m)$ be as in (*). Define l_i, for $i = 1, \cdots, m-1$, as follows: $l_1 = 1$, $l_{i+1} = l(m, l_i)$. Choose a model M of T' such that $|Q^M| \geq 2$, $|Q^M| \geq l_m$, $r = |Q^M|^{n_2} < \aleph_0$, where $n_2 = 2m^{n_0}n_1$ and $|P^M| \geq r^{l_m}$. Let $X_m = P^M$. For $k = 0, \cdots, m-1$ we will define X_{m-k} satisfying the hypothesis of Claim 2 and such that $|X_{m-k}| \geq r^{l_{m-k-1}}$. Suppose X_{m-k_0} satisfying the hypothesis of induction has been found. Let F be the m-place function from X_{m-k_0} into subsets of X_{m-k_0} with less than r elements obtained by letting $F(a_1, \cdots, a_m)$ be a complete set of representatives of the equivalence relation $\sim$ mod $\{a_1, \cdots, a_m\}$. (This relation has at most $|Q^M|^{n^2}$ equivalence classes.) Now by (*) there is a set $B = X_{m-k_0-1}$ with at least $r^{l_{m-k_0-1}}$ elements such that if $a_1, \cdots, a_{m+1} \in X_{m-k_0-1}$ are distinct, then $a_{m+1} \notin F(a_1, \cdots, a_m)$, so a choice of a'_{m+1} to satisfy the hypothesis of Claim 2 can be made from $F(a_1, \cdots, a_m)$.

Now to prove Theorem 1 we need only show

Claim 4. (*) holds.

Remark. Maybe this claim has already appeared in Erdös and Hajnal [3].

Proof. Let $\{y_1, \cdots, y_{r^k}\}$ be random variables on A. What is the probability that $B = \{y_1, \cdots, y_{r^k}\}$ will *not* fulfill the demands of (*)? It is $\leq$

$$\sum_{\substack{i_1,\cdots,i_{m+1}\leq r^k \\ \text{distinct}}} \left[\begin{array}{l}\text{the probability that } y_{\sigma(i_{m+1})} \in F(y_{\sigma(i_1)}, \cdots, y_{\sigma(i_m)}) \\ \text{for some permutation } \sigma \text{ of } i_1, \cdots, i_{m+1}\end{array}\right]$$

$$+ \sum_{1\leq i\neq j\leq r^k} \left[\begin{array}{l}\text{the probability that} \\ y_i = y_j\end{array}\right]$$

$$\leq \binom{r^k}{m+1}\frac{(m+1)!r}{r^l} + \binom{r^k}{2}\frac{1}{r^l} \leq \frac{r^{km+k+1}f(m, k)}{r^l}$$

where $f(m, k)$ is some function of m and k. So we certainly can choose $l = l(m, k)$ such that the whole expression is < 1 for all $r > 1$. This means that it is possible to find a suitable set $\{y_1, \cdots, y_{r^k}\}$. Q.E.D.

This completes the proof of Theorem 1.

4. **Remarks and generalizations.** We now turn to the original problem of the consequences of T having a model of type $(\aleph_\omega, \aleph_0)$. Consider the following combinatorial assertion.

(**) For all $k, m < \omega$ there is $l < \omega$ such that for any m-place function F from $\aleph_l$ to the countable subsets of $\aleph_l$, there is $A \subseteq \aleph_l$, $|A| = \aleph_k$, such that for all distinct $a_1, \cdots, a_m, a_{m+1} \in A$, $a_{m+1} \notin F(a_1, \cdots, a_m)$.

This is the problem mentioned in the introduction; the combinatorial lemma (**) is known to be true for $m = 1$, but for $m > 1$ and even $k = 0$ it is still an open question. See Hajnal [4].

Theorem 2. *If (**) holds and T has a model of type $(\aleph_\omega, \aleph_0)$ then for all λ, μ such that $|T| \leq \mu \leq \lambda < \mathrm{Ded}^*(\mu)$, T has a model of type (λ, μ).*

Proof. As in the proof of Theorem 1 it suffices to show that for all n $\Gamma'_L({}^n2)$ is consistent. To see this let $l = l(k, m)$ be as in (**).

For all $i = 1, \cdots, m-1$ define l_i as follows: $l_1 = 1$, $l_{i+1} = l(l_i, m)$. Now let M be a model of T of type $(\aleph_\omega, \aleph_0)$. For $i = 1, \cdots, m$ we define $A_i \subset P^M$ by retrograde induction, such that $|A_i| = \aleph_{l_i}$: Choose A_m to be any subset of P^M of power $\aleph_{l_m}$. Now assume that A_{i+1} is defined and for all $a_1, \cdots, a_m \in A_{i+1}$ let $F(a_1, \cdots, a_m)$ be a set of representatives in A_{i+1} of each equivalence class of $\sim$ (mod $\{a_1, \cdots, a_m\}$). It is not hard to see that there are $\leq \aleph_0$ such classes; so $|F(a_1, \cdots, a_m)| \leq \aleph_0$, and by (**) there is $A_i \subseteq A_{i+1}$, $|A_i| = \aleph_{l_i}$, such that for all distinct $a_1, \cdots, a_m, a_{m+1} \in A_i$, $a_{m+1} \notin F(a_1, \cdots, a_m)$. The sequence $A_1, \cdots, A_m$ satisfies the requirements of the X_i in Claim 2, and so $T \cup \Gamma'_L({}^n2)$ is consistent. Q.E.D.

We may be interested in other theorems of the form: $\{(m_i, n_i): i < \omega\} \rightarrow (\lambda, \mu)$. Vaught's and Chang's two-cardinal theorems (see e.g. [2]) can easily be generalized to this case, but give less than our result (only when $\lambda \leq \mu^+$, $\mu = \sum_{\kappa < \lambda} \mu^\kappa$). Vaught's two cardinal theorem for cardinals far apart generalizes easily to finite hypothesis (using Ramsey's theorem instead of the Erdös-Rado partition theorem) and it cannot be improved. The following remains open (there are, of course, many others):

Question 1. Is our result best possible? That is, does there exist a sentence for which every n has a model M, $\aleph_0 > |P^M| > |Q^M|^n$, $|Q^M| \geq n$, but does not have a $(2^\mu, \mu)$-model for some μ, and even: has a (λ, μ)-model iff $\mu \leq \lambda < \mathrm{Ded}^*(\mu)$ (assuming for some μ, $\mathrm{Ded}^*(\mu) \leq 2^\mu$).

Conjecture 2. $\{(m_i, n_i, k_i): i < \omega\} \rightarrow (\lambda, \mu, \kappa)$ when $m_i \geq n_i^i$, $n_i \geq k_i^i$, $k_i \geq i$, $\kappa \leq \mu \leq \lambda < \mathrm{Ded}^*\kappa$.

Conjecture 3. $\{(2^{n_i}, n_i): i < \omega\} \rightarrow (2^\mu, \mu)\ [n_i \geq i]$.

The following remarks on the properties of $\Gamma_L(D)$ may be useful:

If in Definition 4, we demand only that $k_{i,j} = \min\{l: \eta_i(l)\} \neq \eta_j(l)\} = \min\{l: \nu_i(l) \neq \nu_j(l)\}$, and $\eta_l(k_{i,j}) = \nu_l(k_{i,j})$, $\eta_j(k_{i,j}) = \nu_j(k_{i,j})$, we get that the consistency of $T \cup \Gamma_L({}^\omega 2)$ implies T has a $(2^\lambda, \lambda)$-model for every λ.

It can be shown that the existence of a model of T of type $(\lambda, \aleph_0)$, where λ is real-valued measurable, implies the consistency of $\Gamma_L({}^\omega 2)$, even for sentences of $L_{\omega_1, \omega}$.

Papageorgiou shows that our method gives a positive answer to Conjecture 2 if we strengthen the assumption to: $k_i \geq i$, $n_i \geq (k_i)^i$, $m_i \geq (n_i)^{(n_i)^i}$; and that this generalizes to any finite number instead of three.

It is trivial that if T has a model M, $|P^M| \geq \aleph_0 > |Q^M|$, then for every

$\lambda \geq |T|$, T has a model of type $(\lambda, |Q^M|)$. Also if for every n, T has a model M, $|P^M| \geq \aleph_0 > |Q^M| \geq n$, then for every $\lambda \geq \mu \geq |T|$, T has a model of type (λ, μ). Hence in Theorem 1 we ignore those cases.

On n-cardinal theorems see Chang and Keisler [2]. Our result was announced in [5], and [6, § 0, (6) p. 251]. In [6, § 0] there is a discussion on n-cardinal problems.

Added in proof. The main conjecture has been proved and submitted to the Proceedings of the American Mathematical Society.

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