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THE MEASURE OF THE INTERSECTION OF ROTATES OF A SET ON THE CIRCLE[1]

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ABSTRACT. Let S be a set of real numbers modulo 1 of Lebesgue measure less than 1. It is shown that for every $\epsilon > 0$ and for large k, there exist translates $S + y_1, \cdots, S + y_k$ of S such that the measure of their intersection is less than ϵ^k.

1. Let U be the group of real numbers modulo 1, and S a subset of Lebesgue measure $\mu(S) < 1$. Given real numbers $y_1, \cdots, y_k$, write $\mu(y_1, \cdots, y_k)$ for the measure of the intersection of the k translates $S + y_1$, $S + y_2, \cdots, S + y_k$. Finally, denote by $\phi(k)$ the infimum of $\mu(y_1, \cdots, y_k)$ over all k-tuples $y_1, \cdots, y_k$. Erdős, Rubel and Spencer[2] had conjectured that

$$\lim_{k\to\infty} \phi(k)^{1/k} = 0. \tag{1}$$

In the present note we shall prove this conjecture.

The convergence expressed by (1) is not uniform with respect to the sets S. In fact, it can be shown that for $0 < \alpha < 1$, $\epsilon > 0$ and $k \geq 1$, there exist sets S with $\mu(S) = \alpha$ and $\phi(k)^{1/k} > \alpha - \epsilon$.

2. Since $\mu(S) < 1$, the set S is contained in a countable union of intervals whose total measure is less than 1. In fact, this is true even with intervals of the type $a \leq x < b$[3] with rational endpoints a, b. Hence we may assume that S itself is a countable union of such intervals.

Using the easily established relation

$$\int_U \mu(y_1, \cdots, y_m, z_1 + x, \cdots, z_n + x)\, dx = \mu(y_1, \cdots, y_m)\mu(z_1, \cdots, z_n),$$

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[2] P. Erdős, L. A. Rubel and J. H. Spencer, in the problem collection of the 1972 number theory conference in Colorado.

[3] We are considering intervals modulo 1. Hence if $\{x\}$ denotes the fractional part of x, the interval $a \leq x < b$ consists of numbers x modulo 1 with $\{x - a\} < \{x - b\}$.

one sees that $\phi(m+n) \leq \phi(m)\phi(n)$. Hence if t is any positive integer, we have $\phi(jt) \leq \phi(t)^j$ $(j = 1, 2, \cdots)$, and if k is a large integer with $jt < k \leq (j+1)t$, then $\phi(k) \leq \phi(jt)\phi(k - jt) \leq \phi(jt) \leq \phi(t)^j$ and

$$\phi(k)^{1/k} \leq \phi(t)^{j/k} \leq \phi(t)^{(1/t)-(1/k)}.$$

Therefore the limit superior of $\phi(k)^{1/k}$ as $k \to \infty$ cannot exceed $\phi(t)^{1/t}$. Thus in order to prove (1), it will suffice to show that for every $\epsilon > 0$ there is an integer t with

(2) $$\phi(t)^{1/t} < \epsilon.$$

3. Write $\mu(S) = \mu$, and choose $\delta > 0$ so small that

(3) $$2\delta < 1 - \mu \quad \text{and} \quad (\delta/(1-\mu-\delta))^{1-\mu-\delta} < \epsilon.$$

We may write $S = S_1 \cup S_2$, where S_1 is a finite union of intervals $a \leq x < b$ with rational endpoints, and where $\mu(S_2) < \delta$.

Let r be a common denominator of the endpoints of the intervals contributing to S_1. Choose an integer s with $s > 1/\delta$, and put

(4) $$t = rs, \qquad \nu = 1/t.$$

Let $\chi(x)$ be the characteristic function of S, and write

$$I_\nu(y) = \int_y^{y+\nu} \chi(x)\, dx.$$

Lemma. *The function*

$$J_\nu(z) = I_\nu(z+\nu) I_\nu(z+2\nu) \cdots I_\nu(z+t\nu)$$

satisfies $J_\nu(z) \leq (\epsilon\nu)^t$.

To prove the Lemma, we observe that S_1 consists of a finite number (in fact less than r) intervals E of the type $(u/r) \leq x < (u+1)/r$ with integral u. For each such interval E contained in S_1, let E' be the enlarged interval $(u/r) - (1/t) \leq x < (u+1)/r$. Let S_1' be the union of the intervals E' so obtained. It is clear that

(5) *if* $x + w \in S_1$ *with* $0 \leq w \leq \nu$, *then* $x \in S_1'$.

For each interval E above we have $\mu(E') = \mu(E) + (1/t)$, and hence we have $\mu(S_1') < \mu(S_1) + (r/t) = \mu(S_1) + (1/s) < \mu(S) + \delta = \mu + \delta$. Now S_1' is a disjoint union of intervals $(v/t) \leq x < (v+1)/t$ with integral v. If, say, it is a disjoint union of p such intervals, then $\mu(S_1') = p/t$ and hence

(6) $$p = t\mu(S_1') < t(\mu + \delta).$$

Exactly $q = t - p$ of the numbers $z + \nu, z + 2\nu, \cdots, z + t\nu$ lie outside S_1'; let these be the numbers $z + m_1\nu, z + m_2\nu, \cdots, z + m_q\nu$. Since each integral $I_\nu(y)$ is always $\leq \nu$, we have

$$J_\nu(z) \leq \nu^p I_\nu(z + m_1\nu) \cdots I_\nu(z + m_q\nu). \tag{7}$$

Now $I_\nu(z + m_i\nu)$ is the integral of $\chi(x)$ over the interval $z + m_i\nu \leq x < z + (m_i + 1)\nu$ $(i = 1, \cdots, d)$. These intervals are disjoint from each other. Furthermore, since $z + m_i\nu \notin S_1'$, (5) implies that these intervals are disjoint from S_1. Therefore if $\bar{S}_1$ is the complement of S_1, we have

$$I_\nu(z + m_1\nu) + \cdots + I_\nu(z + m_q\nu) \leq \int_{\bar{S}_1} \chi(x)\,dx \leq \mu(S_2) < \delta.$$

By the arithmetic-geometric inequality, the product of the q integrals on the left is $< (\delta/q)^q$, and (7) yields

$$J_\nu(z) < \nu^p(\delta/q)^q = \nu^t(\delta t/q)^q.$$

From (6) we have $q = t - p > t(1 - \mu - \delta)$, whence

$$(\delta t/q)^q < (\delta/(1 - \mu - \delta))^q < (\delta/(1 - \mu - \delta))^{t(1-\mu-\delta)} < \epsilon^t$$

by (3), and the Lemma is proved.

4. The desired inequality (2) follows at once from the Lemma by observing that

$$\begin{aligned}
\phi(t) &\leq \nu^{-t} \int_\nu^{2\nu} dy_1 \cdots \int_{t\nu}^{(t+1)\nu} dy_t\, \mu(-y_1, \cdots, -y_t) \\
&= \nu^{-t} \int_U dx \int_\nu^{2\nu} dy_1 \cdots \int_{t\nu}^{(t+1)\nu} dy_t\, \chi(x + y_1) \cdots \chi(x + y_t) \\
&= \nu^{-t} \int_U dx\, I_\nu(x + \nu) I_\nu(x + 2\nu) \cdots I_\nu(x + t\nu) \\
&= \nu^{-t} \int_U J_\nu(x)\,dx < \nu^{-t}(\epsilon\nu)^t = \epsilon^t.
\end{aligned}$$

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