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ON THE p-ELEMENTS OF A FINITE GROUP

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ABSTRACT. Let G be a finite group, p a prime, and x a p-element in G. An element g in G is called a witness of G if the subgroup generated by x and g is a p-group. The set of all witnesses of x in G is denoted by $W(x)$. This paper shows that x belongs to a given Sylow p-subgroup P of G if one of the following holds: (1) G is p-solvable and $W(x) \supset P \cap \{x^g | g \in G\}$; (2) G is p-solvable, $P = \langle P \setminus Z(P) \rangle$, and $W(x) \supset P \setminus Z(P)$; (3) $\mathrm{cl}(P) \le 2$ and $W(x) \supset P$; (4) x normalizes a subgroup P_1 of P with $|P: P_1| \le p^2$ and $W(x) \supset P$; (5) $|P| = p^4$ and $W(x) \supset P$.

1. Introduction and notation. Let G be a finite group, p a prime number, and x a p-element in G. An element g in G is called a witness of x if the subgroup generated by x and g is a p-group. The set of all witnesses of x in G is denoted by $W(x)$.

It is interesting to know how $W(x)$ reflects the property of x. For example, it is interesting to see which property of $W(x)$ will imply that x belongs to the maximal normal p-subgroup of G, and which property of $W(x) \cap H$ will imply that x belongs to H, where H is a given p-subgroup of G.

When $W(x)$ contains the conjugacy class of x, Baer's theorem tells us that x belongs to the maximal normal p-subgroup of G.

This paper will show that x belongs to a given Sylow p-subgroup when certain conditions are imposed on the Sylow p-subgroup and $W(x)$. This is summarized in

Theorem 1. *Let G be a finite group and P a Sylow p-subgroup of G. For $x \in G$, if one of the following holds, then $x \in P$.*

(1) *G is p-solvable and $W(x) \supset P \cap \{x^g | g \in G\}$.*

(2) *G is p-solvable, $P = \langle P \setminus Z(P) \rangle$, and $W(x) \supset P \setminus Z(P)$.*

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(3) $\mathrm{cl}(P) \leq 2$ *and* $W(x) \supset P$.

(4) x *normalizes a subgroup* P_1 *of* P *with* $|P: P_1| \leq p^2$ *and* $W(x) \supset P$.

(5) $|P| = p^4$ *and* $W(x) \supset P$.

Some examples are given in the final section which show that the conclusion of Theorem 1 becomes invalid when the stated conditions are relaxed.

All groups in this paper are assumed to be finite. Our notation is standard and taken mainly from [1]. In particular, let G be a group. Then $\mathrm{Soc}(G)$, $\mathrm{Sol}(G)$, $O_p(G)$, $F(G)$, $\Phi(G)$, $\exp(G)$ and G' denote respectively the joint of all minimal normal subgroups, the largest solvable normal subgroup, the maximal normal p-subgroup, the Fitting subgroup, the Frattini subgroup, the exponent and the commutator subgroup of G. Moreover, for any nilpotent group G, $\mathrm{cl}(G)$ denotes the nilpotent class of G.

2. **Proof of Theorem 1.** Suppose X is an arbitrary finite group. If $\mathrm{Sol}(x) = 1$, then it is well known that $C_X(\mathrm{Soc}(X)) = 1$. This can be seen from the following. Since $\mathrm{Sol}(X) = 1$, $\mathrm{Soc}(X) = S_1 \times \cdots \times S_n$, where each S_i is a nonabelian group. Thus $C_X(\mathrm{Soc}(X)) \cap \mathrm{Soc}(X) = 1$. However, $C_X(\mathrm{Soc}(X)) \lhd X$ which, together with the fact that $\mathrm{Sol}(X) = 1$, implies $C_X(\mathrm{Soc}(X)) = 1$.

We use induction on $|G|$ to prove Theorem 1; assume that Theorem 1 is false. Hence $O_p(G) = 1$ and $G = \langle P, x \rangle$. Let K be the conjugacy class containing x and let S be any minimal normal subgroup of G. By induction we can see that $G = PS = SP$. Let $B(x) = \{X | X = \langle x, A \rangle$ is a p-group and $A \supset P\}$. Choose $R \in B(x)$ such that $|R \cap P|$ is of maximal order.

Cases (1) *and* (2). Since G is p-solvable and $O_p(G) = 1$, S is a normal subgroup of order relative prime to p. Let $x^s \in K \cap P$ for some $s \in S$. If $P \cap K \subset W(x)$, then $\langle x^s, x \rangle$ is a p-group. However $x^{-1}x^s = [x, s] \in S$. Therefore $W(x) \supset P \cap K$ implies $[x, s] = 1$, and so $x = x^s \in P$. This proves (1).

Now suppose Case (2) occurs. For $g \in G$, we have $g = a(g) \cdot b(g)$, where $a(g) \in S$ and $b(g) \in P$. Hence $x^g \in P \setminus Z(P)$ if and only if $x^{a(g)} \in P \setminus Z(P)$. Since $[x, a(g)] = x^{-1}x^{a(g)} \in S$, $K \cap P \subset Z(P)$. Hence $[x^s, y] = 1$ for all $y \in P$. As

$$[x^s, y] = [x[x, s]y] = [x, y]^{[x,s]}[[x, s], y],$$

$[x, y] = ([[x, s], y]^{-1})^{[x,s]^{-1}}$. Since $S \lhd G$ and $W(x) \supset P \setminus Z(P)$, $[x, y] = 1$ for all $y \in P \setminus Z(P)$. As $\langle P \setminus Z(P) \rangle = P$, $[x, P] = 1$, which is impossible. This contradiction completes the proof of Case (2).

Case (3). By (1) we may assume that $\mathrm{Sol}(G) = 1$. Hence $\mathrm{Soc}(G) = S$. Let $F = \{[y, x] \mid y \in P\}$. If $F = 1$, then $x \in P$. Therefore $1 \neq \langle F \rangle \subset C_G(x)$. For any $y, z \in P$, we have $[y, x]^z = [yz, x][z, x]^{-1}$. This shows $P \subset N_G(\langle F \rangle)$. Hence $\langle F \rangle \lhd G$. Since $\mathrm{Sol}(G) = 1$, $C_G(\mathrm{Soc}(G)) = 1$. Thus $S \subset \langle F \rangle$. Now $x \in C_G(S)$ implies $x = 1$. Of course $1 \in P$, a contradiction. This completes the proof of (3).

Case (4). The case $|P: P_1| = p$ is trivial. Hence we may assume that $|P: P_1| = p^2$. Therefore there exists an element $y \in N_P(P_1) \backslash P_1$ such that $\langle x, y \rangle$ is a p-group. Since $\langle x, y \rangle \subset N_G(P_1)$, $\langle x, y, P_1 \rangle \in B(x)$. So $|R \cap P| \gneq |P_1|$. This shows that $R \cap P$ is a maximal subgroup of any Sylow p-subgroup containing it. Clearly x normalizes $R \cap P$ and an application of (3) completes the proof of (4).

Case (5). By (3) and (4) we may assume that $\mathrm{cl}(P) > 2$ and $|R \cap P| \not> p^2$. Therefore $P' \not\subset Z(P)$, $P' = \Phi(P)$, and $P/P' \cong Z_p \times Z_p$. Suppose $\exp(P) = p^2$. Let $y \in P$ be an element of order p^2 and let $D = \langle y \rangle$. We may assume that $R \cap P = D$. Suppose $x \in N_G(D)$. Let $y_1 \in N_P(D) \backslash D$. Then $\langle x, D, y_1 \rangle \in B(x)$, which violates the maximality of R. Therefore $x \notin N_G(D)$, and $R = \langle y, x \rangle$ is a Sylow p-subgroup of G. Let $N = N_R(D)$. Then $N \neq R$ and $N \supsetneq D$. Since $N^x = N$ and $|N| = p^3$, $D \cap D^x = E$, where E is the unique subgroup of order p in D. So x centralizes E. If $D \lhd P$, then $E \subset Z(P)$, which implies $E \subset Z(G)$, a contradiction. So $D \ntriangleleft P$. Let $N_1 = N_P(D)$ and let $u \in P \backslash N_1$. Then $N_1 = N_1^u$ and so $D \cap D^u = E$. This implies $E \subset Z(P)$ which is impossible. Hence $\exp(P) = p$. If $p = 2$, then P is abelian, a contradiction. Hence p is odd. Let A be a maximal abelian subgroup of P containing P'. If $A \neq P'$, then $A \cong Z_p \times Z_p \times Z_p$, and $P = A\langle y \rangle$ with $y^P = 1$. We can view A as a 3-dimensional vector space over the finite field of p-elements, and y a linear transformation on A. With respect to a suitable basis $\{v_1, v_2, v_3\}$ of A, y^{-1} has the matrix representation

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $v_1^{y^{-1}} = v_1 v_2$ and $v_2^{y^{-1}} = v_2 v_3$. So

$$\begin{aligned}(yv_1)^p &= (yv_1y^{-1})(y^2v_1y^{-2}) \cdots (y^{(p-1)}v_1y^{-(p-1)})y^{p-1} \cdot yv_1 \\ &= v_1^{y^{-1}} \cdot v_1^{y^{-2}} \cdots v_1^{y^{-(p-1)}} \cdot v_1.\end{aligned}$$

Let $z = y^{-1}$. Using the linear transformation notation, from $z = 1$, we see that $v_1^{(z-1)^p} = 1$. Therefore

$$1 = v_1^{z+z^2+\dots+z^{p-1}+1} = (v_1^{(z-1)})^{-1} = v_2^{-1}.$$

This is impossible. Hence P' is a maximal abelian subgroup of P. So $P' = C_P(P')$, and P/P' is isomorphic to a subgroup of $\mathrm{Aut}(P')$. Since the Sylow p-subgroup of $\mathrm{Gl}(2, P)$ has order p, $|P| = p^3$, a contradiction. This completes the proof of (5).

Remark. The proof of (5) also shows that a p-group P of order p^4 with $\exp(P) = p$ is not of maximal class.

A stronger version of (3) in Theorem 1 is the following.

Proposition 1. *Let H be a nilpotent subgroup of a finite group G and let $x \in G$. If for any $t \in H$, $\langle x, t\rangle$ is a nilpotent group of nilpotent class not greater than 2, then $\langle H, x\rangle$ is nilpotent.*

Proof. We apply induction to $|G|$. We may assume that $G = \langle H, x\rangle$. Let S be a minimal normal subgroup. By induction, G/S is nilpotent. Set $F = \{[t, x] \mid t \in H\}$. For any $t_1, t_2 \in H$ we have $[t_1, x]^{t_2} = [t_1 t_2, x][t_2, x]^{-1}$. Therefore $\langle F\rangle \lhd G$ and $x \in C_G\langle F\rangle$. If $F = \{1\}$, then the conclusion of the proposition holds. Hence we may assume $F \neq \{1\}$.

Suppose $\mathrm{Sol}(G) = 1$. Then $S = \mathrm{Soc}(G)$. Since $C_G(\mathrm{Soc}(G)) = 1$ and $\langle F\rangle \lhd G$, $S \subset \langle F\rangle$. Therefore $x \in C_G(S)$, and so $x = 1$. Of course $\langle H, x\rangle$ is a nilpotent group in this case.

If $\mathrm{Sol}(G) \neq 1$, then we may assume that S is an elementary abelian p-group. Therefore G is solvable. By induction we may assume that $S = F(G)$. By [1, p. 218], $C_G(S) = S$. Therefore $S \subset \langle F\rangle$, and so $x \in S$. Hence $G = HS$. For any prime number $q \neq p$, x centralizes every q-element of H. Since H is nilpotent, H is a p-group. Hence G is a p-group. This completes the proof of Proposition 1.

3. **Some examples.** If we remove the condition $\langle P \backslash Z(P)\rangle = P$ in (2) of Theorem 1, then any G with more than one nontrivial abelian Sylow p-subgroup will be a counterexample. The following is a less trivial example in this direction.

Example (a). Let V be a 3-dimensional vector space over the field F with four elements. Choose a basis for V which we use to identify the elements in $\mathrm{Gl}(V)$ by its representing matrices. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} f & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $f \in F \backslash \{0, 1\}$ and $f^3 = 1$. Then $A^3 = B^3 = 1$ and $\langle A, B \rangle$ is a 3-group. Let $G = V\langle A, B \rangle$ be the semidirect product of V and A, B such that for $v \in V$ and $T \in \langle A, B \rangle$, v^T is defined to be $v + v(T - 1)$. In G we use the multiplicative notation. Let

$$C = B^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

After a short calculation we see that

$$C_V(C) \cap [(C_V(A) \backslash C_V(C)) \cdot (C_V(B) \backslash C_V(C))]$$

is not empty. Let $v_1 \in C_V(A) \backslash C_V(C)$, $v_2 \in C_V(B) \backslash C_V(C)$ and $v_1 v_2 \in C_V(C)$. Hence $1 = [v_1 v_2, C^{-1}] = [v_1, C^{-1}]^{v_2} [v_2, C^{-1}]$. Since V is an elementary abelian 2-group, $[v_1, C^{-1}] = [v_2, C^{-1}]$. Let $v = [v_1, C^{-1}] \neq 1$. Then $\langle vC, C \rangle$ is not a 3-group. But $\langle vC, A \rangle^{v_1} = \langle vC^{v_1}, A \rangle = \langle v[v_1, C^{-1}]C, A \rangle = \langle C, A \rangle$ is a 3-group. Similarly $\langle vC, B \rangle^{v_2} = \langle C, B \rangle$ is a 3-group. This shows that although $\langle vC, A \rangle$, $\langle vC, B \rangle$ are 3-groups and $P = \langle A, B \rangle$ is a Sylow 3-subgroup of G, $vC \notin \langle A, B \rangle$. Of course G is 3-solvable. Since $Z(\langle A, B \rangle) = \langle BB^A B^{A^2} \rangle$, $P = \langle P \backslash Z(\langle A, B \rangle) \rangle$. Also $|P| = 3^4$ in this example. Therefore it can be used to show that if we replace the stated condition in (5) by $W(x)$ which contains some set of generators of P, then the conclusion is false.

Example (b). Let p be a prime and let F be the finite field with p elements. Let G be the group of all 3×3 invertible matrices with determinant 1 over F.

Suppose

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $G = \langle x, y, z \rangle$ and $P = \langle y, z \rangle$ is a Sylow p-subgroup of G. Clearly $\mathrm{cl}(P) = 2$. It is not difficult to see that $\langle x, y \rangle$, $\langle x, z \rangle$ are both p-groups. Of course $x \notin P$. This shows that if we just require that $\langle x, y \rangle$ is a p-group for y belonging to a given set of generators of P, then the conclusion of (2) might be false.

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