PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 1, March 1975

CONTINUOUS METRIC PROJECTIONS

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ABSTRACT. An example is given of a reflexive, rotund Banach space whose dual space is not Fréchet differentiable such that every metric projection onto closed subspaces is norm continuous. This example shows that several published conjectures on necessary and sufficient conditions for a reflexive, rotund Banach space to have norm continuous metric projections onto all closed subspaces are incorrect.

Introduction. If X is a reflexive, rotund Banach space, every closed subspace of X is a unique best approximation subspace, called a Chebyshev subspace. If M is a closed subspace of X, the metric projection P_M associated with M is defined via $\inf_{m \in M} \|x - m\| = \|x - P_m(x)\|$. An open question in best approximation theory is to find necessary and sufficient conditions on a reflexive, rotund Banach space to insure that every closed subspace has a norm continuous metric projection associated with it. In [5], the conjecture was that every reflexive and rotund Banach space had this property. In [4], this was later amended to be that Banach spaces with a Fréchet smooth dual space had this property. In [4], it was also conjectured that if the canonical duality map from X^* to X, restricted to $M^\perp = \{x^* \in X^* | x^*(m) = 0, \forall m \in M\}$, was norm continuous, then P_M was norm continuous. It was verified for subspaces M of finite codimension in [4]. In [1], the study of bounded compactness led to the continuity of P_M for those M such that ker $P_M = \{x \in X | P_M(x) = \theta\}$ was boundedly compact. Further, if the codimension of M was finite, then the bounded compactness of ker P_M was a necessary and sufficient condition that P_M be norm continuous.

In this paper, we show that there exists a reflexive, rotund Banach space whose dual is not Fréchet smooth and yet every metric projection is norm continuous. This is accomplished by closely examining the space X exhibited by Klee in [6], where X is a renorm of l_2, which is reflexive, Gâteaux smooth at all points of the unit ball except $\{-\delta_0, \delta_0\}$. By well-known duality relationships, X^* is a reflexive, rotund Banach space. It is this space which will be

Received by the editors September 29, 1972.
AMS (MOS) subject classifications (1970). Primary 41A65; Secondary 46B10.
Key words and phrases. Abstract approximation, metric projections, best approximation maps.

shown to have the desired properties. In §1 we will recall the Klee space, following the notation in [6]. In §2 we recall the terminology of abstract approximation and show that the dual of the Klee space provides an example which exhibits a reflexive, rotund Banach space whose dual is not Fréchet smooth and yet every metric projection onto closed subspaces is norm continuous. In §3, we show that the theorems in [4] and [1] cannot be extended when the dimension requirements on M are removed.

In this paper X will denote a Banach space and X^* its continuous dual. $U(X)$ and $S(X)$ will denote the closed unit ball in X and its boundary, respectively. All other notation will correspond to that in [2]. A source for the theory of metric projections is [3, §32].

1. **The Klee space.** A renorm of l_2 is given in [6] as follows. An element x is in l_2 if and only if $x = \{x_i\}_{i=0}^{\infty}$ such that $\Sigma_{i=0}^{\infty}|x_i|^2 < \infty$. Let $V = \{x \in l_2 \,|\, x_0 = 0\}$. Let U_V and S_V denote the unit ball and its boundary, respectively, of the subspace V. For each bounded sequence $a = \{a_i\}_{i=1}^{\infty}$, let T_a denote the linear transformation of V into V defined by $T_a(x) = (0, a_1x_1, a_2x_2, \cdots)$, $x \in V$. For each $\lambda \in [-1, 1]$ and each sequence $\eta = (\eta_1, \eta_2, \cdots)$ of even functions on $[-1, 1]$ to $[0, 1]$ with $\eta_i(0) = 1$ for all i, let $\eta(\lambda) = (\eta_1(\lambda), \eta_2(\lambda), \cdots)$. Then let

$$U_\eta = \bigcup_{|\lambda|\le 1} [\lambda\delta_0 + T_{\eta(\lambda)}U_V] \quad \text{and} \quad S_\eta = \bigcup_{|\lambda|\le 1} [\lambda\delta_0 + T_{\eta(\lambda)}S_V]$$

where $\delta_0 = (1, 0, 0 \cdots)$. Further let U be the convex closure of U_η and let S be the boundary of U. Klee prescribed the following conditions on $\{\eta_i\}$ to obtain the required smoothness conditions.

K1. η_i is continuous and concave, with $\eta_i(0) = 1$, $\eta_i(1-\epsilon_i) = 2\epsilon_i$ and $\eta_i(1) = 0$ for all i.

K2. η_i is differentiable on $[0, 1]$ with $\eta_i'(0) = 0$ and $\eta_i'(1-\epsilon_i) = -1$ for all i.

K3. η_i has a vertical tangent at 1, i.e. $\lim_{\lambda\to 1}\eta_i'(\lambda) = -\infty$ for all i.

K4. $\{\epsilon_i\}_{i=1}^{\infty}$ is a sequence of real numbers such that $\epsilon_i \in [0, 1/6)$ with $\epsilon_i \to 0$ as $i \to \infty$.

We wish to place a further restriction on the η_i to facilitate computational problems.

K5. For all $i, \eta_i(\lambda) \ge (1-\lambda^2)^{1/2}$.

The gauge of U, ρ_U is the renorm of l_2. The remaining facts to be recalled for our investigation are:

Ka. $U_\eta = \{-\delta_0, \delta\} \cup \{x \in l_2 \,|\, |x_0| < 1 \text{ and } \Sigma_{i=1}^{\infty}(x_i/\eta_i(x_0))^2 \le 1\}$.

Kb. The intersection of S_η with the half plane $G_s = L + [0, \infty)s$, $s \in S_V$, $L = \{\lambda\delta_0 \,|\, -\infty < \lambda < \infty\}$, is given by $\{\lambda\delta_0 + r_s(\lambda)\cdot s \,|\, |\lambda| \le 1\}$, where $r_s(\lambda)$

is the positive solution of $\Sigma_{i=1}^{\infty}[r_s(\lambda)s_i/\eta_i(\lambda)]^2 = 1$.

Kc. The Klee space (X, ρ_U) is reflexive, Gâteaux smooth at all points of U and Fréchet smooth at all points of U except $\{-\delta_0, \delta_0\}$.

2. **The example.** Recall that if M is a closed subspace of a Banach space Z, $d(z, M) = \inf\{\|z - m\| \mid m \in M\}$. Let $P_M(z) = \{m \in M \mid \|z - m\| = d(z, M)\}$. The set valued mapping $z \to P_M(z)$ is called the metric projection of Z onto M. If P_M is a single valued mapping, M is called a Chebyshev subspace. The set $\{z \in Z \mid P_M(z) = \theta\}$ will be denoted $\ker P_M$.

Let $(Y, \|\cdot\|)$ denote $(X, \rho_U)^*$ where (X, ρ_U) is the Klee space described in §1. Since (X, ρ_U) was reflexive and smooth, one has $(Y, \|\cdot\|)$ is reflexive and rotund. Let M be any Chebyshev subspace of Y. Let ψ_M: $Y \backslash M \to S(\ker P_M)$ via $\psi_M(y) = (y - P_M(y))/\|y - P_M(y)\|$. Further, let T: $Y^* \to Y$ be the canonical duality given by $Ty^* = \{y \in Y \mid y(y^*) = \|y\| \, \|y^*\|\}$. Theorem 13 in [14] states that if T is continuous at $f \in S(Y^*) \cap M^{\perp}$, then P_M is continuous at all points of the set $\psi_M^{-1}(T(f))$. Since $Y^* = X$, the Klee space T is continuous at all points of $S(Y^*)$ except $\{-\delta_0, \delta_0\}$. Since the continuity of P_M at all points is equivalent to the continuity of P_M at all points in $\ker P_M$, we need only study those points x_0 and those subspaces M, such that $(x_0 - P_M x_0)/\|x_0 - P_M x_0\| \in T(\pm\delta_0)$, since P_M would be continuous at all other points. In particular, $\delta_0 \in M^{\perp}$ for such x_0 and M. Thus if $\delta_0 \notin M^{\perp}$, P_M is norm continuous. We wish to show every P_M is norm continuous. Thus we consider those subspaces M such that $\delta_0 \in M^{\perp}$. We use the fact that since (X, ρ_U) is a renorm of l_2, $(Y, \|\ \|)$ can be considered a renorm of l_2.

Lemma 2.1. *Given M, a Chebychev subspace of Y such that $\delta_0 \in M^{\perp} \subset X$, then $M^{\perp}$ taken as a set in Y is contained in* $\ker P_M$.

Proof. Let z be in $S(M^{\perp}) \subset Y$. One must show that z is in $\ker P_M$. Now z acts as a linear functional on $X = Y^*$ and since X is reflexive, z attains its norm on U. In particular, by use of the Krein-Milman theorem, z must attain its norm at an extreme point and hence at a point of S_η. Thus there exists $u \in S_\eta$ such that $(u, z) = 1$. If $u \in S_\eta \cap M^{\perp} \subset X$ then $z \in Tu \subset \ker P_M$, since $T(M^{\perp}) = \ker P_M$, and the result follows. If for all $u \in S_\eta \cap M^{\perp}$, $(u, z) \neq 1$, choose any $u \in S_\eta$ such that $(u, z) = 1$. By previous remarks in §1, u has the form $u = \lambda\delta_0 + r_y(\lambda)y$, $y \in s_V$, $\lambda \neq 1$. But y can be written as $y = x + w$, with x in $M^{\perp}$ and w in M considered as sets in X since X is a renorm of l_2. Since $\delta_0 \in M^{\perp} \subset X$, $M \subset V \subset Y$. Thus w is in V and hence x is in V. Now,

$$(u, z) = (\lambda\delta_0 + r_y(\lambda)(x + w), z) = (\lambda\delta_0 + r_y(\lambda)x, z)$$

since z is in $M^{\perp}$. We claim $\rho_U(\lambda\delta_0 + r_y(\lambda)x) < 1$. This follows since

$$1 = \rho_U(u) = \sum_{i=1} \left[\frac{r_y(\lambda)y_i}{\eta_i(\lambda)}\right]^2 = \sum_{i=1} \frac{r_y^2(y)(x_i + w_i)^2}{\eta_i(\lambda)^2}$$

$$= \sum_{i=1} r_y^2(y) \frac{x_i^2 + 2x_iw_i + w_i^2}{\eta_i(\lambda)^2} = \sum_{i=1} r_y^2(\lambda) \frac{x_i^2 + w_i^2}{\eta_i(\lambda)^2}.$$

The last inequality follows from

$$\left|\sum_{i=1} \frac{x_iw_i}{\eta_i(\lambda)^2}\right| \leq \left|\frac{\Sigma x_iw_i}{1 - \lambda^2}\right| = 0$$

since $(x, w) = \Sigma x_iw_i = 0$ because $x \in M^{\perp} \cap V$, $w \in M \cap V$, and **K5**. Thus

$$\sum_{i=1} \frac{r_y^2(\lambda)x_i^2}{\eta_i(-1)^2} = 1 - \sum_{i=1} \frac{r_y^2(\lambda)w_i^2}{\eta_i(\lambda)^2} = k.$$

Since u is not an element of $M^{\perp}$, $w \neq \theta$, and hence $k < 1$. Thus $\rho_U(\lambda\delta_0 + r_y(\lambda)x) = k < 1$. We remark that if $k = 0$ then $x = \theta$ and $\lambda = 1$, implying $z = \delta_0$ and $u = \delta_0$, a contradiction. By considering $\alpha = k^{-1}(\lambda\delta_0 + r_y(\lambda)x)$ one has $\rho_U(\alpha) = 1$ and $\alpha \in M^{\perp}$. But $(\alpha, z) = k^{-1}(\lambda\delta_0 + r_y(\lambda)x, z) = k^{-1} > 1$. This contradicts $\|z\| = 1$. Thus there exists $u \in S_\eta \cap M^{\perp}$ such that $z \in Tu$. Therefore, $M^{\perp} \subset \ker P_M$. Q.E.D.

By [5, Theorem 3], if $\ker P_M$ contains a subspace N such that $M + N = Z$, the entire Banach space, then P_M: $Z \to M$ is linear and hence continuous. Since Y is a renorm of l_2, one can write $Y = M + M^{\perp}$. The above remarks coupled with Lemma 2.1 yield that P_M is continuous whenever $\delta_0 \in M^{\perp}$. The initial remarks in §2 showed that P_M is continuous whenever $\delta_0 \notin M^{\perp}$. Hence, we have the following

Theorem 2.1. *There exists a reflexive, rotund Banach space Y such that Y^* is not Fréchet smooth, but P_M is norm continuous for all closed subspaces M.*

This example shows that the conjecture in [4], that the necessary and sufficient condition for a reflexive, rotund Banach space to have the metric projections onto all closed subspaces continuous, be that the dual space be Fréchet smooth, is incorrect.

3. **Further applications as counterexamples.** In [4], Theorem 14 states: If X is a reflexive, rotund Banach space, M a Chebyshev subspace of finite codimension and if T is the canonical duality map $T\colon X^* \to X^{**} = X$, then $T/M^\perp$ is norm continuous if and only if P_M is norm continuous.

It was conjectured that this theorem would also be valid without any dimension requirements on M. The following example shows that this cannot be the case.

Example 3.1. Let X be the Klee space, Y its dual as above. Let $M \subset Y$ be defined as $M = \text{closed span}\{\delta_{2j} \mid j = 1, 2, \cdots\}$ where δ_j are the usual basis vectors in l_2. Clearly the closed span of $\{\delta_0, \delta_{2j+1} \mid j = 0, 1, 2 \cdots\}$ is contained in $M^\perp$. Consider the sequence $\{x_{2i+1}\}$ in $S(M^\perp)$ defined via $x_{2i+1} = (1 - \epsilon_{2i+1})\delta_0 + 2\epsilon_{2i+1}\delta_{2i+1}$ where the $\{\epsilon_i\}$ are as in **K4** in §1. It is easily seen that $x_{2i+1} \to \delta_0$. By an elementary calculation $T(x_{2i+1}) = (\delta_0 + \delta_{2i+1})/(1 + \epsilon_{2i+1})$. Hence $T(x_{2i+1})$ converges weakly to δ_0, but not in norm. Hence $T/M^\perp$ is not continuous, but by the work of the previous section P_M is linear and continuous.

A set is boundedly compact if every bounded sequence has a convergent subsequence. In [1], Theorem 8 states: If X is a normed linear space, M a Chebyshev subspace of finite codimension, then $\ker P_M$ is boundedly compact if and only if P_M is norm continuous.

It was conjectured that the theorem continued to be valid without restriction on the dimension of M. Using Example 3.1, we can show that the theorem cannot be extended. The set $\{T(x_{2i})\}_{i=0}^\infty$ consists of elements of norm one in $\ker P_M$. However, there does not exist a norm convergent subsequence, and yet, P_M is norm continuous.

It should be noted that B. Kripke of Ohio State University, in an unpublished result, has found an example of a Hilbert space which can be renormed with a rotund norm so as to contain a closed subspace M such that P_M is not norm continuous.

Finally in [7], Ošman has announced necessary and sufficient conditions for all metric projections to be continuous. These conditions resemble conditions implying Fréchet smoothness of the dual space but are slightly weaker. The example in §2 gives a concrete example of such a space.

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