PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 1, March 1975

A REMARK ON THE NONLINEAR COMPLEMENTARITY PROBLEM

GEORGE LUNA

ABSTRACT. It is shown that previous sufficient conditions for the solution of the nonlinear complementarity problem can be considerably weakened using a result of Rockafellar.

Let E be a Banach space and K a closed convex cone in E. The *polar* K° of K is the set $\{x^* \in E^* | \sup x^*(K) \leq 0\}$. The *indicator function* ψ_K for K is defined for each $x \in E$ by $\psi_K(x) = 0$ if $x \in K$ and by $\psi_K(x) = \infty$ if $x \notin K$. The subdifferential $\partial\psi_K$ of ψ_K is the set-valued map defined for each $x \in E$ by $\partial\psi_K(x) = \{x^* \in E^* | x^* \in K^\circ$ and $\langle x, x^* \rangle = 0\}$. A map $T: E \to 2^{E^*}$ is *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $x^* \in T(x)$, $y^* \in T(y)$ and $x, y \in D(T) = \{x | T(x) \neq \emptyset\}$. We say that T is α*-monotone* if there exists a strictly increasing function $\alpha: [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$ such that $\langle x - y, x^* - y^* \rangle \geq \|x - y\| \alpha(\|x - y\|)$ whenever $x^* \in T(x)$ and $y^* \in T(y)$ for each $x, y \in D(T)$. If $\alpha(r) = kr$ for $k > 0$, then T is said to be *strongly monotone*. If T is single valued and continuous from line segments in $D(T)$ to E^* with the weak* topology, then T is said to be *hemicontinuous*.

We consider the following version (to multivalued maps) of the generalized complementarity problem (GCP) as formulated by Karamardian [2]:

Let $T: E \to 2^{E^*}$; find $x \in K$ and $x^* \in T(x)$ satisfying $x^* \in -K^\circ$ and $\langle x, x^* \rangle = 0$.

The (GCP) is important in that it is the form for many problems in mathematical programming, game theory, economics, etc. For this we refer the reader to [2] and [4] and the references given there.

It is our purpose in this note to prove the following theorem which has as a corollary an extension of a recent theorem of Bazaraa, Goode, and

---

Received by the editors September 5, 1973 and, in revised form, October 9, 1973 and January 7, 1974.

AMS (MOS) subject classifications (1970). Primary 46N05, 47H05, 47H15; Secondary 90C30.

Key words and phrases. Nonlinear complementarity problem, monotone operators, nonlinear programming.

Nashed [1]. The proof is almost an immediate consequence of the following result of Rockafellar [5, Proposition 2].

Proposition. *Let* E *be reflexive, and let* $F\colon E \to E^*$ *be a maximal monotone operator. Suppose there exists a* $\beta > 0$ *such that* $\langle x, x^* \rangle \geq 0$ *whenever* $\|x\| > \beta$, $x \in D(F)$, $x^* \in F(x)$. *Then there exists an* $x \in E$ *such that* $0 \in F(x)$.

Some criteria for maximal monotonicity are mentioned in Remark 1 at the end of the article. These are due to Rockafellar [5].

Theorem. *Let* E *be a reflexive Banach space and* K *a closed convex cone in* E. *Suppose* $T\colon E \to 2^{E^*}$ *and* $T + \partial\psi_K$ *is maximal monotone. In addition, suppose*

$$(*) \qquad \begin{array}{l} \text{there exists } \beta > 0 \text{ such that } \langle x, x^* \rangle \geq 0 \\ \text{whenever } x \in D(T) \cap K,\ \|x\| > \beta \text{ and } x^* \in T(x). \end{array}$$

Then the (GCP) *has a solution.*

Proof. Suppose $x^* \in (T + \partial\psi_K)(x)$, where $x \in D(T) \cap K$ and $\|x\| > \beta$; then $x^* = y^* + z^*$, where $y^* \in T(x)$ and $z^* \in \partial\psi_K(x)$. Thus

$$\langle x, x^* \rangle = \langle x, y^* + z^* \rangle = \langle x, y^* \rangle + \langle x, z^* \rangle = \langle x, y^* \rangle$$

since $z^* \in \partial\psi_K(x)$ and K is a cone implies that $\langle x, z^* \rangle = 0$. Thus from $(*)$ we have $\langle x, x^* \rangle \geq 0$. Hence the hypotheses of the Proposition are satisfied for $T + \partial\psi_K$, and there exists $x \in K$ such that $0 \in (T + \partial\psi_K)(x)$. This is easily seen to be equivalent to the conclusion of the Theorem.

As a corollary we obtain the following extension of the theorem of Bazaraa et al. [1]. In [1], the operator T is assumed to be bounded.

Corollary. *Let* E *be a reflexive Banach space and* K *a closed convex cone in* E. *Suppose* $T\colon K \to E^*$ *is hemicontinuous and* α*-monotone. Then there exists a unique solution to the* (GCP).

Proof. By a well-known result of Browder and Stampacchia [5, Theorem 3], $T + \partial\psi_K$ is maximal monotone. Since T is α-monotone we have

$$\langle x, T(x) \rangle \geq \|x\|\alpha(\|x\|) + \langle x, T(0) \rangle \geq \|x\|(\alpha(\|x\|) - \|T(0)\|).$$

Choose β large enough so that $\|x\| > \beta \Rightarrow \alpha(\|x\|) \geq \|T(0)\|$; then $(*)$ is satisfied and the existence part of the Corollary follows from the Theorem. The uniqueness is immediate from the definition of α-monotonicity.

Remark 1. When K is a convex cone and T is maximal monotone, three sufficient conditions (see [5]) for $T + \partial\psi_K$ to be maximal monotone are:

(1) $\dim E < \infty$ and $(\operatorname{ri} D(T)) \cap K \neq \emptyset$ (ri C denotes interior with respect to the affine hull of C).

(2) $D(T) \cap \operatorname{int} K \neq \emptyset$.

(3) $\operatorname{int} D(T) \cap K \neq \emptyset$.

In the main theorem of Karamardian [3], E is finite dimensional and K is the nonnegative orthant. From (1) it is easily seen that Karamardian's hypothesis that T be continuous is not needed.

Remark 2. Condition (*) in the Theorem is easily seen to be satisfied by all maps previously considered in the context of the (GCP), e.g. strongly monotone, copositive, etc.; cf. [2].

Remark 3. Bazaraa et al., remarked in [1] that they were unable to generalize the proofs of Karamardian by replacing Kakutani's fixed point theorem with recent fixed point theorems of F. E. Browder and Ky Fan. We note that the proof of Rockafellar's result uses that $R(T + J) = E^*$, where T is monotone and J is the duality map. This is a result of Browder which uses one of his fixed point theorems.

The author would like to thank R. T. Rockafellar for valuable comments on an earlier version of this note. He would also like to thank R. R. Phelps for his help and encouragement.

REFERENCES

1. M. S. Bazaraa, J. J. Goode and M. Z. Nashed, *A nonlinear complementarity problem in mathematical programming in Banach space*, Proc. Amer. Math. Soc. 35 (1972), 165–170. MR 45 #9211.

2. S. Karamardian, *Generalized complementarity problem*, J. Optimization Theory Appl. 8 (1971), 161–167.

3. ———, *The nonlinear complementarity problem with applications*. I, II, J. Optimization Theory Appl. 4 (1969), 87–98; ibid. 4 (1969), 167–181. MR 40 #5261.

4. C. E. Lemke, *Recent results on complementarity problems*, Nonlinear Programming (Proc. Sympos. Univ. of Wisconsin, Madison, Wis., 1970), Academic Press, New York, 1970, pp. 349–384. MR 42 #7281.

5. R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. 149 (1970), 75–88. MR 43 #7984.

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE POLYTECHNIC UNIVERSITY, POMONA, CALIFORNIA 91768

Current address: Department of Mathematics, University of California, San Diego, La Jolla, California 92037