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STIEFEL-WHITNEY NUMBERS AND MAPS COBORDANT TO EMBEDDINGS

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ABSTRACT. A necessary and sufficient condition is given for a continuous map between compact differentiable manifolds to be cobordant in the sense of Stong to an embedding. For the case of a map $f: M^n \longrightarrow S^{n+k}$ the condition reduces to the vanishing of all Stiefel-Whitney numbers of M^n that involve $\bar{w}_i$ for $i \geq k$.

1. **Introduction.** A necessary condition for the existence of an embedding of a compact differentiable manifold M^n in a euclidean space R^{n+k} (or a sphere S^{n+k}) is the vanishing of the dual Stiefel-Whitney classes $\bar{w}_i(M^n) \in H^i(M^n; Z/2Z)$ for $i \geq k$. This condition is far from sufficient. For example, if M^n is a real projective n-space P^n with $n = 2^s - 1$ $(s \geq 4)$, then $\bar{w}_i(P^n) = 0$ for all $i > 0$, but P^n does not embed in R^{n+k} if $k < n/4$. (See [1, p. 131].) However, one can still look for some statement involving embeddings that is implied by the condition $\bar{w}_i(M^n) = 0$ for $i \geq k$. Because Stiefel-Whitney numbers form a complete system of invariants for certain cobordism theories one can expect a result involving cobordism, and in [2] we have shown that if $\bar{w}_i(M^n) = 0$ for $i \geq k$ then M^n is cobordant to a manifold M_1^n that embeds in S^{n+k} provided that k is not much smaller than n. Equivalently, under the same conditions, any map $f: M^n \longrightarrow S^{n+k}$ is *bordant* to an embedding $f_1: M_1^n \longrightarrow S^{n+k}$.

In this paper we use the notion of cobordism of maps due to Stong and prove that the vanishing of all Stiefel-Whitney numbers of M^n involving $\bar{w}_i(M^n)$ $(i \geq k)$ is necessary and sufficient for a map $f: M^n \longrightarrow S^{n+k}$ to be *cobordant as a map* to an embedding $f_1: M_1^n \longrightarrow N_1^{n+k}$. In other words if you weaken the original embedding problem using cobordism of maps, then the whole story is told by the usual dual Stiefel-Whitney numbers.

Actually we solve the more general problem of determining when a map $f: M^n \longrightarrow N^{n+k}$ is cobordant to an embedding $f_1: M_1^n \longrightarrow N_1^{n+k}$. In the next

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section we develop some necessary conditions involving Stiefel-Whitney numbers of f, and in the third section we state the theorems and prove sufficiency of our conditions. Here the proof is based on a construction suggested to me by Stong. The last section is devoted to examples and remarks. Throughout we use homology and cohomology with $Z/2Z$ coefficients.

2. **Stiefel-Whitney numbers of maps.** Given a map $f\colon M^n \to N^{n+k}$, we have induced maps f^* and f_* in cohomology. Recall that if $x \in H^i(M^n)$ then $f_*(x) = D_N f_*(x \cap [M]) \in H^{i+k}(N^{n+k})$, where D_N denotes Poincaré duality for N^{n+k}, and where f_* is also used to denote the induced homology map of f. Then Stong [5] shows that the cobordism class of f is completely determined by the Stiefel-Whitney numbers of f, namely the numbers

$$\langle w_\omega(N) \cdot f_* w_{\omega_1}(M) \cdot \ldots \cdot f_* w_{\omega_r}(M), [N]\rangle.$$

Here $[N]$ denotes the fundamental homology class in $H^{n+k}(N)$, $\omega = (i_1, \cdots, i_p)$, $w_\omega = w_1^{i_1} \cdot \ldots \cdot w_p^{i_p}$, $|\omega| = i_1 + \cdots + i_p$, and $|\omega| + \Sigma_{j=1}^r(|\omega_j| + k) = n + k$.

We find it convenient to rewrite those numbers with $r > 0$ so as to have classes in $H^*(M)$ evaluated on $[M]$. Observe that

$$\langle a \cdot f_*(b) \cdot f_*(c), [N]\rangle = \langle a \cdot f_*(b), f_*(c) \cap [N]\rangle = \langle a \cdot f_*(b), f_*(c \cap [M])\rangle$$

$$= \langle f^*(a) \cdot f^* f_*(b), c \cap [M]\rangle = \langle f^*(a) \cdot f^* f_*(b) \cdot c, [M]\rangle.$$

Thus the numbers of f with $r > 0$ take the form

$$\langle f^* w_\omega(N) \cdot f^* f_* w_{\omega_1}(M) \cdot \ldots \cdot f^* f_* w_{\omega_{r-1}}(M) \cdot w_{\omega_r}(M), [M]\rangle. \tag{1}$$

Now suppose that f is an embedding with normal bundle ν. Then $f^* f_*(a) = a \cdot w_k(\nu)$. (This is because f_* has another interpretation, namely, $f_* = c^*\Phi$, where $c\colon N \to T(\nu)$ is the collapsing map of N onto the Thom space of ν, and $\Phi\colon H^*(M) \to H^{*+k}(T(\nu))$ is the Thom isomorphism. If $i\colon M \to T(\nu)$ is the inclusion of the zero section then $f^* f_*(a) = f^* c^* \Phi(a) = i^* \Phi(a)$, and finally $i^*\Phi(a) = a \cdot w_k(\nu)$ by a basic property of the Thom isomorphism.) To see how $w_k(\nu)$ is determined by f, let N be embedded in a euclidean space R^t with normal bundle η. Then $\tau M^n \oplus \nu \oplus f^{-1}\eta$ is a trivial bundle, so $w(M)w(\nu)f^*\overline{w}(N) = 1$, and hence $w(\nu) = \overline{w}(M)f^*w(N)$. Note that $w_i(\nu) = 0$ if $i > k$ because ν is a k-dimensional bundle.

Definition. $\overline{w}(f)$ for any map $f\colon M^n \to N^{n+k}$ is defined by $\overline{w}(f) = \overline{w}(M)f^*w(N)$.

We now have a necessary condition for f to be cobordant to an embedding, namely that $\bar{w}_i(f)$ should be zero in numbers if $i > k$, and that the numbers of the form (1) should be equal to the numbers of the form (2) below:

$$(2) \qquad \langle f^* w_\omega(N) \cdot w_{\omega_1}(M) \cdot \ldots \cdot w_{\omega_r}(M) \cdot (\bar{w}_k(f))^{r-1}, [M] \rangle .$$

3. The main results.

Theorem. *A map $f: M^n \to N^{n+k}$ $(k > 0)$ is cobordant to an embedding $f_1: M_1^n \to N_1^{n+k}$ if and only if the following conditions hold:*

(i) *All Stiefel-Whitney numbers of f involving $\bar{w}_i(f)$ for $i > k$ are zero.*

(ii) *All Stiefel-Whitney numbers of f as given by* (1) *are equal to the corresponding Stiefel-Whitney numbers as given by* (2).

Proof. We have just shown that (i) and (ii) are necessary, so now let f be a map that satisfies (i) and (ii). We wish to construct a cobordant embedding f_1. If t is large, the map $(f, 0)$: $M^n \to N^{n+k} \times \mathbf{R}^t$ is homotopic to an embedding with normal bundle η classified by a map $\bar{\eta}$: $M^n \to BO$. Because $\tau M \oplus \eta \simeq f^{-1}\tau N \oplus t\epsilon$, we see that $w(\eta) = \bar{w}(M) \cdot f^* w(N) = \bar{w}(f)$. It follows that the Stiefel-Whitney numbers of the map $\bar{\eta}$ which are used to determine the bordism class of this map are a subset of the Stiefel-Whitney numbers of the map f, and the condition (i) of the hypotheses implies that all Stiefel-Whitney numbers of $\bar{\eta}$ involving $w_i(\eta)$ for $i > k$ are zero. Hence $\bar{\eta}$ is bordant to a map that factors through $BO(k)$, say $\bar{\eta}_1$: $M_1^n \to BO(k) \subset BO$ with associated bundle η_1 over M_1^n. (See [3, 17.3, p. 48].) Let $S(\eta_1 \oplus 1)$ denote the sphere bundle of $\eta_1 \oplus 1$ over M_1, let $N_1^{n+k} = S(\eta_1 \oplus 1) \cup N^{n+k}$, and let f_1: $M_1^n \to N_1^{n+k}$ be the inclusion of the cross-section determined by the trivial line bundle. (I wish to thank R. E. Stong for showing me this construction in the case where $N^{n+k} = S^{n+k}$.) Then f_1 is a differentiable embedding, and it remains to show that f and f_1 are cobordant. For this purpose we compute the Stiefel-Whitney numbers of f_1 and compare them with those of f.

Because of the section, $H^*(M_1^n)$ is a direct summand of $H^*(S(\eta_1 \oplus 1))$ so the Serre spectral sequence for $H^*(S(\eta_1 \oplus 1))$ collapses and $H^*(S(\eta_1 \oplus 1))$ is isomorphic to $H^*(M_1^n) \oplus H^*(S^k)$. If p: $S_1 = S(\eta_1 \oplus 1) \to M_1$ is the projection, then $\tau S_1 = p^{-1}\tau M_1 \oplus \phi$, where ϕ is the bundle along the fibres. Thus $f_1^{-1}\tau N_1 = \tau M_1 \oplus f_1^{-1}\phi = \tau M_1 \oplus \eta_1$, and we obtain the equations

$$f_1^* w(N_1) = w(M_1) \cdot w(\eta_1) \quad \text{and} \quad w(\eta_1) = \bar{w}(M_1) f^* w(N_1) = \bar{w}(f_1).$$

Now we are ready to compare numbers. First the numbers of f and f_1 with $r = 0$ both equal the numbers of N because $S(\eta_1 \oplus 1)$ is a boundary.

If $r > 0$, the numbers of f and f_1 of the form (1) reduce to those of the form (2) by hypothesis (ii) for f and by construction (i.e., f_1 is an embedding) for f_1. Now we use the fact that $f^*w(N) = w(M) \cdot \overline{w}(f) = w(M)w(\eta)$ and the analogous fact for f_1 to rewrite the number of the form (2) into the form $\langle w_\omega(M)w_{\omega'}(\eta), [M]\rangle$ with an analogous expression for f_1. But now we are looking at Stiefel-Whitney numbers of $\overline{\eta}$ and of $\overline{\eta}_1$ and these are equal because $\overline{\eta}$ and $\overline{\eta}_1$ are bordant maps. This completes the proof.

Corollary. *A map $M^n \to S^{n+k}$ $(k > 0)$ is cobordant to an embedding f_1: $M^n \to N_1^{n+k}$ if and only if all Stiefel-Whitney numbers of M^n involving $\overline{w}_i(M^n)$ $(i \geq k)$ are zero. (In interpreting the statement of the Corollary it helps to note that all maps f: $M^n \to S^{n+k}$ $(k > 0)$ are cobordant.)*

Proof. Taking $N^{n+k} = S^{n+k}$ in the Theorem we find that $w(S^{n+k}) = 1$ and that $f^*f_*(x) = 0$ for all $x \in H^*(M)$. Thus condition (i) is equivalent to saying that all Stiefel-Whitney numbers involving $\overline{w}_i(M^n)$ $(i > k)$ should vanish, and condition (ii) is then equivalent to saying that all numbers involving $\overline{w}_k(M^n)$ should vanish.

4. **Applications and examples.** If we apply the Corollary to a product we obtain the following result.

Proposition 1. *If maps $M^m \to S^{m+p}$, $N^n \to S^{n+q}$ are both cobordant to embeddings, then any map $M^m \times N^n \to S^{m+n+p+q-1}$ is cobordant to an embedding. In other words, products always embed better (modulo map corbordism) than the product embedding of the factors. We assume $p > 0$, $q > 0$.*

Proof. This follows from the Corollary because the top nonzero class (in numbers) of $M^m \times N^n$ is $\overline{w}_{p+q-2}(M^m \times N^n) = \overline{w}_{p-1}(M^m) \cdot \overline{w}_{q-1}(N^n)$.

Remark. One can ask whether the proposition is true without the "modulo map cobordism" clause. In many cases it does hold. For let $d(X)$ denote the difference between the best euclidean immersion and best euclidean embedding of the manifold X. If $d(M^m) > 0$ and $m \leq n + p + q - 2$, then we can embed $M^m \times N^n$ in $R^{m+n+p+q-1}$ given embeddings of M^m in R^{m+p} and N^n in R^{n+q}. (See [4, p. 319].) But by [4, pp. 320, 321] the product embedding of $(CP^2)^2$ is best possible. However this manifold is cobordant to $(RP^2)^4$ whose product embedding is not best. Hence "modulo map cobordism" cannot be deleted but might possibly be improved to "modulo bordism".

Now consider the case $M^n = P^n$, a real projective n-space. Recall that if α generates $H^1(P^n)$ then $H^*(P^n) = (Z/2Z)[\alpha]/(\alpha^{n+1})$ and $w(P^n) =$

$(1+\alpha)^{n+1}$. If n is odd then all Stiefel-Whitney numbers of P^n are zero and a map $P^n \longrightarrow S^{n+k}$ $(k>0)$ is cobordant to the obvious embedding $S^n \subset S^{n+k}$. So let n be even and write $n = 2^a + b$ with $0 \leq b < 2^a$.

Proposition 2. *A map $f\colon P^n \longrightarrow S^{n+k}$ (n even, $k>0$) is cobordant to an embedding if and only if $k \geq n-2b$.*

Proof. $\bar{w}(P^n) = (1+\alpha)^{-n-1} = (1+\alpha)^{-2^{a+1}}(1+\alpha)^{2^a-b-1} = (1+\alpha^{2^{a+1}})^{-1}(1+\alpha)^{2^a-b-1} = (1+\alpha)^{2^a-b-1}$. Let $p = 2^a - b = n-2b$. Then $\bar{w}_i(P^n) = 0$ if $i \geq p$ but the Stiefel-Whitney number $w_1^{n-p+1}\bar{w}_{p-1} \neq 0$.

Remark. The extreme cases are $n = 2^a$ and $n = 2^{a+1}-2$. In the first case we get $k \geq n$ and this shows that a high codimension may be needed even for embeddings modulo map cobordism. In the second case we get $k \geq 2$. An example of a codimension 2 embedding f_1 cobordant to f may be constructed as follows. Let $M_1^n = P^n$, let $N_1^{n+2} = P^{n+1} \times S^1$, and let f_1 be the inclusion of P^n into $P^{n+1} \times \{1\}$. Then the normal bundle of f_1 admits a section, so $\bar{w}_2(f_1) = 0$, and hence $f_1^* f_{1*}(x) = 0$ for all x. Also $w(N_1^{n+2}) = 1$. Thus the Stiefel-Whitney numbers of f_1 reduce to those of P^n and the same is true for the Stiefel-Whitney numbers of f.

Proposition 3. *Let $f\colon P^n \longrightarrow P^{n+k}$ $(k>0)$ be a map. If $f^*(\alpha) \neq 0$ then f is cobordant to the inclusion $P^n \subset P^{n+k}$. If $f^*(\alpha) = 0$ then f is cobordant to an embedding if and only if n is odd or $k \geq n-2b$ where $n = 2^a + b$ as above.*

Proof. If $f^*(\alpha) = \alpha$ (with the obvious abuse of notation) then f has the same Stiefel-Whitney numbers as the inclusion $P^n \subset P^{n+k}$. If $f^*(\alpha) = 0$ then $f^* f_*(x) = 0$ for all x, and $f^* w(P^{n+k}) = 1$. Thus f has the same Stiefel-Whitney numbers as a map $P^n \longrightarrow S^{n+k}$ and Proposition 2 applies.

Finally we mention the homotopy theoretic interpretation of our results. Stong [5] has shown that cobordism classes of maps from n-manifolds to $(n+k)$-manifolds are in 1-1 correspondence with the bordism group $\mathfrak{N}_{n+k}(\Omega^s MO(k+s))$, s large. On the other hand, $\mathfrak{N}_{n+k}(MO(k))$ represents cobordism classes of embeddings of codimension k. The obvious map $BO(k) \longrightarrow BO(k+s)$ yields a map $\Sigma^s MO(k) \longrightarrow MO(k+s)$ and hence a map $\Psi\colon MO(k) \longrightarrow \Omega^s MO(k+s)$. The induced homomorphism $\mathfrak{N}_*(\Psi)$ on bordism is injective. (For the Stiefel-Whitney numbers of a map $\phi\colon N^{n+k} \longrightarrow MO(k)$ can be written in terms of the associated embedding $f\colon M^n \longrightarrow N^{n+k}$ and take the form (2) of §2. They are included among the Stiefel-Whitney numbers of f considered now as a map.) We have described the image of $\mathfrak{N}_*(\Psi)$ in terms of computable invariants. There is another problem whose solution must also be given by Stiefel-Whitney numbers, namely finding which bordism classes of maps $f\colon M^n \to N^{n+k}$ can be represented by an embedding $f_1\colon M_1^n \to N^{n+k}$. In terms of homotopy theory we are trying to describe the image of the set of homotopy classes $[N^{n+k}, MO(k)]$ in the group $[N^{n+k}, \Omega^s MO(k+s)] = \mathfrak{N}^k(N^{n+k}) = \mathfrak{N}_n(N^{n+k})$. (See [3, p. 37].) The appropriate Stiefel-Whitney numbers of f have the form $\langle w_\omega(M) f^*(x), [M] \rangle$, where $x \in H^*(N)$ need not be a characteristic class of N. In the case where $N^{n+k} = S^{n+k}$, these numbers reduce to the numbers of M^n and it would be interesting to know the answer here and to see whether new nonembedding theorems could be deduced using Stiefel-Whitney classes.

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