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A NOTE ON LOCALLY FINITE GROUP ALGEBRAS

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ABSTRACT. We obtain an injectivity condition for group algebras which is equivalent to local finiteness.

1. **Introduction.** Several authors have studied the effect of various injectivity conditions on group algebras. Connell [1] showed that if the group algebra $F[G]$ is self-injective, then G is locally finite; Renault [4] improved this result by showing that G is, in fact, finite. The following question arises: what weakening of self-injectivity coincides with local finiteness? This note provides one answer.

We will say that a ring R with 1 is *principally (right) self-injective* if any right R-module map from a principal right ideal of R into R can be lifted to all of R. Notice that this definition is the usual Baer criterion for self-injectivity if we omit the two occurrences of the word "principal".

If M is a right R-module and S is a subset of R, then $l_M(S) = \{m \in M | ms = 0 \ \forall s \in S\}$ is the *left annihilator* of S in M. Left actions give rise to right annihilators. If $R = M$ we say that a left ideal L of R is a left annihilator if $L = l_R(S)$ for some subset $S \leq R$; equivalently, $L = l_R(r_R(L))$. We will drop subscripts when the context is clear.

If G is a group and F is a field, then $F\{G\}$ will denote the set of all infinite formal sums $\Sigma f_g g$ with $f_g \in F$ and $g \in G$. Under pointwise addition $F\{G\}$ becomes a right $F[G]$-module containing $F[G]$.

Finally we can state our result.

Theorem. *The following properties are equivalent*:

1. $F[G]$ *is principally self-injective.*
2. G *is locally finite.*
3. $F\{G\} \cdot a \cap F[G] = F[G] \cdot a \quad \forall a \in F[G]$.

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4. *Every principal left ideal of* $F[G]$ *is an annihilator.*

The equivalence of 2 and 4 is of particular interest. It might be regarded as a first approximation to the following longstanding conjecture: if every element of $F[G]$ is a zero-divisor or invertible then G is locally finite.

2. **A proof.** Crucial to all proofs of local finiteness is

Lemma [3, p. 105]. *Let* $g_1, \cdots, g_n$ *be a finite number of elements of* G, *and let* $H = \langle g_1, \cdots, g_n \rangle$ *be the subgroup of* G *they generate. Then*

$$\{r \in F[G] \mid (g_i - 1)r = 0 \text{ for } i = 1, \cdots, n\} = \begin{cases} 0 & \text{if } H \text{ is infinite,} \\ (\Sigma_{h \in H} h)F[G] & \text{if } H \text{ is finite.} \end{cases}$$

We proceed to the Theorem:

$1 \Longrightarrow 2$. It suffices to prove that $\langle H, x \rangle$ is finite whenever H is a finite subgroup of G and $x \in G$. (One can then argue local finiteness by inducting on the number of generators of a finitely generated subgroup of G.) Set $s = \Sigma_{h \in H} h$. The Lemma shows either $\langle H, x \rangle$ is finite or $(x-1)sa = 0$ $\Longrightarrow sa = 0$ $\forall a \in F[G]$. In the latter case the $F[G]$-map ϕ: $(x-1)sF[G] \longrightarrow F[G]$ given by $((x-1)sa)\phi = sa$ is well defined. By hypothesis $\exists\, d \in F[G]$ $\ni s = ((x-1)s)\phi = d(x-1)s$, i.e. $(1 - d(x-1))s = 0$. Since both $d(x-1)$ and any annihilator of s are in the augmentation ideal of $F[G]$, 1 is in the augmentation ideal, a contradiction. Thus $\langle H, x \rangle$ is finite.

$2 \Longrightarrow 3$. Let $\{t_i\}_{i \in I}$ be a left transversal for the finite subgroup $\langle \text{supp } a \rangle = H$ in G. If $(\Sigma_{i \in I} t_i b_i) \cdot a \in F[G]$ with $b_i \in F[H]$ then $S = \{i \in I \mid b_i \cdot a \neq 0\}$ is finite. Since H is finite, $\Sigma_{i \in S} t_i b_i \in F[G]$ and $(\Sigma_I t_i b_i) \cdot a = (\Sigma_S t_i b_i) \cdot a$.

$3 \Longrightarrow 4$. It is enough to show that $l_{F\{G\}}(r_{F[G]}(a)) = F\{G\}a$. The inclusion "$\supseteq$" is trivial. If b is in the double annihilator then the F-linear map $\tau: aF[G] \longrightarrow F$ given by $\tau(ar) = \operatorname{tr}(br)$ is well defined. (Here, tr of an element in $F\{G\}$ denotes the coefficient of 1.) Lift τ to an F-linear map on $F[G]$. Writing $a = \Sigma_h a_h h$, a finite sum, we have

$$b = \sum_{g \in G} \operatorname{tr}(bg^{-1})g = \sum_g \tau(ag^{-1})g = \sum_g \tau\left(\sum_h a_h hg^{-1}\right)g$$

$$= \sum_g \left(\sum_h a_h \tau(hg^{-1})\right)g = \sum_h \left(\sum_g \tau(hg^{-1})gh^{-1}\right)a_h h = \left(\sum_{\gamma \in G} \tau(\gamma^{-1})\gamma\right)a \in F\{G\}a.$$

$4 \Longrightarrow 1$. It is easy to see that any ring R is principally right self-injective iff for each $a \in R$, $Ra = l(r(a))$ [2, Theorem 1].

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