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# AUTOMORPHISM GROUPS OF ABELIAN $p$-GROUPS

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ABSTRACT. Let $\Gamma$ be the automorphism group of a nonelementary reduced abelian $p$-group, $p \geq 5$. It is shown that every noncentral normal subgroup of $\Gamma$ contains a noncentral elementary abelian normal $p$-subgroup of $\Gamma$ of rank at least 2.

1. **The result.** Throughout the following, $G$ is a reduced $p$-primary abelian group, $p \geq 5$, and $\Gamma$ is the group of all automorphisms of $G$.

If $G$ is elementary abelian then the normal structure of $\Gamma$ is well known. In particular, $\Gamma$ does not contain normal $p$-subgroups $\neq 1$ [2, pp. 41, 45]. If $pG \neq 0$ then $\Gamma$ does possess nontrivial normal $p$-subgroups. Moreover, in this case, every noncentral (i.e. not contained in the center $Z\Gamma$ of $\Gamma$) normal subgroup of $\Gamma$ contains a noncentral normal $p$-subgroup $N$ of $\Gamma$ such that $N^p = 1$ [6, Theorem A].

The purpose of this note is to prove the following result which is considerably stronger.

**Theorem.** *Let $\Gamma$ be the automorphism group of a nonelementary reduced abelian $p$-group, $p \geq 5$. Then every noncentral normal subgroup of $\Gamma$ contains a noncentral elementary abelian normal $p$-subgroup of $\Gamma$ of rank at least* 2.

The hypothesis $p \neq 2$ is indispensable since the dihedral group $D_4$ occurs as an automorphism group of an abelian 2-group (namely $G = Z(2) \oplus Z(4)$; $D_4$ contains a [noncentral] cyclic normal subgroup of order 4 whose socle is the center of $D_4$). Whether the above Theorem holds true for $p = 3$ is an open question.

2. **Tools.** Notation and terminology will be that of [3] and [6] unless explained otherwise. Calculations involving automorphisms are carried out in the endomorphism ring of $G$. The following facts are used frequently. Note that mappings are written to the right.

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(2.1) *The center of* $\Gamma$. The center of $\Gamma$ consists precisely of the multiplications with units in the ring $R_p$ of $p$-adic integers [1, pp. 110, 111]. If $G$ is unbounded then $Z\Gamma \simeq R_p^*$ and the center of $\Gamma$ contains no elements of order $p$. An automorphism $\alpha$ of $G$ is central if and only if $S\alpha = S$ for all subgroups $S$ of $G$ [1, pp. 110, 111]. $\Gamma$ is commutative if and only if $G$ is (locally) cyclic [3, p. 222].

(2.2) *Stabilizers.* Let $\operatorname{fix}(B/A)$ be the set of all $\gamma \in \Gamma$ inducing the identity mapping in $B/A$ where $A \leq B$ are subgroups of $G$. If $A$ and $B$ are characteristic in $G$ then $\operatorname{fix}(B/A)$ is a normal subgroup of $\Gamma$. The stabilizer of $A$ in $G$ is defined as $\operatorname{stab} A = \operatorname{fix} A \cap \operatorname{fix}(G/A)$. It is well known that $\operatorname{stab} A \simeq \operatorname{Hom}(G/A, A)$. In particular, stabilizers are abelian.

(2.3) *The normal subgroups of exponent p.* Let $N$ be a normal subgroup of $\Gamma$ such that $N^p = 1$. Then $N \leq \Psi$ where $\Psi$ consists of all $\gamma \in \Gamma$ such that $G[p](\gamma - 1) \leq pG$ and $p(\gamma - 1) = 0$ [4, pp. 409, 410]. If $\psi \in \Psi$ then $G(\psi - 1) \leq G[p]$ and $(\psi - 1)^3 = 0$ [4, p. 411]. Hence $\Psi \leq \operatorname{fix}(pG) \cap \operatorname{fix}(G/G[p])$ and, since $p \geq 3$, $\Psi^p = 1$.

An immediate consequence is the following result.

**Lemma 2.4.** *If* $N$ *is a normal p-subgroup of* $\Gamma$ *such that* $N^p = 1$ *then* $N \cap \operatorname{fix} G[p]$ *and* $N \cap \operatorname{fix}(G/pG)$ *are elementary abelian normal p-subgroups of* $\Gamma$.

**Proof.** By (2.3) $N \leq \Psi$, and $\Psi \cap \operatorname{fix} G[p] \leq \operatorname{stab} G[p]$, $\Psi \cap \operatorname{fix}(G/pG) \leq \operatorname{stab} pG$. By (2.2)

$$\operatorname{stab} G[p] \simeq \operatorname{Hom}(G/G[p], G[p]), \qquad \operatorname{stab} pG \simeq \operatorname{Hom}(G/pG, pG),$$

which are elementary abelian.

The following two lemmas are technical.

**Lemma 2.5.** *Let* $G = A \oplus \langle h \rangle$ *where* $A$ *has rank at least two and* $p^m A = 0 \neq p^m G$ *for some integer* $m \geq 1$. *Let* $N$ *be a normal subgroup of* $\Gamma$ *such that* $N^p = 1$. *If there exists* $\gamma \in N$ *such that* $(\gamma - 1)^2 \neq 0$ *then* $N \cap \operatorname{fix} G[p]$ *is noncentral.*

**Proof.** Let $\gamma = 1 + \tau \in N$ such that $\tau^2 \neq 0$. Then $G\tau \leq G[p]$, $p\tau = 0$ (cf. (2.3)), and $\tau^2 \neq 0$ implies $G\tau \not\leq pG$. Since $G$ is generated by its elements of maximal order, one can assume $h\tau^2 \neq 0$. Hence $h\tau \in G[p] \backslash pG$ and

$$G = \langle a \rangle \oplus B \oplus \langle h \rangle, \qquad h\tau = a, \qquad a\tau \neq 0, \qquad B \neq 0.$$

Suppose $pB = 0$. Then $pG = \langle ph \rangle$ and $\langle p^m h \rangle \geq (pG)[p] \geq G[p]\tau \geq (\langle a \rangle \oplus B)\tau$. Hence $a\tau \neq 0$ and $B \neq 0$ imply

$$\langle a \rangle \oplus B = \langle a \rangle \oplus K, \qquad K\tau = 0 \neq K.$$

In this case, pick any $0 \neq x \in K$. If $pB \neq 0$ pick $0 \neq x \in (pB)[p]$ and put $K = B$. In either case

$$G = \langle a \rangle \oplus K \oplus \langle h \rangle, \qquad 0 \neq x \in K[p], \qquad x\tau = 0.$$

Define the endomorphism $\sigma$ of $G$ by

$$a\sigma = x, \qquad K\sigma = 0, \qquad h\sigma = 0.$$

Then $\sigma^2 = 0$ and $G\sigma\tau = \langle x \rangle \tau = 0$. Lemma 2.6 of [7] implies $\delta = 1 + \tau\sigma \in N$. From $h(\delta - 1) = h\tau\sigma = a\sigma = x \notin \langle h \rangle$ it follows that $\delta \notin Z\Gamma$ (cf. (2.1)). Since

$$G[p](\delta - 1) = G[p]\tau\sigma \leq pG\sigma = p \cdot \langle x \rangle = 0,$$

$\delta \in N \cap \operatorname{fix} G[p]$, completing the proof.

**Lemma 2.6.** *Let $G$ and $N$ be as in Lemma 2.5 and suppose that $(\gamma - 1)^2 = 0$ for all $\gamma \in N$. If $N \cap \operatorname{fix} G[p] \leq Z\Gamma$ and $N \cap \operatorname{fix}(G/pG) \leq Z\Gamma$ then $N \leq Z\Gamma$.*

**Proof.** Assume by way of contradiction that there exists $\gamma = 1 + \tau \in N \backslash Z\Gamma$. Then $\gamma \notin \operatorname{fix}(G/pG)$ and, as above, one can assume $h\tau \notin pG$, consequently $h\tau \in G[p] \backslash pG$ and

$$(2.7) \qquad G = \langle a \rangle \oplus B \oplus \langle h \rangle, \qquad h\tau = a, \qquad O(a) = p, \qquad a\tau = h\tau^2 = 0.$$

By hypothesis $\gamma \notin \operatorname{fix} G[p]$ and hence $G[p]\tau \neq 0$. Since $G[p] \leq \langle a \rangle \oplus B[p] \oplus \langle ph \rangle$ and $a\tau = 0$, $p\tau = 0$, this implies the existence of $b \in B[p] \backslash pB$ such that

$$(2.8) \qquad b\tau \neq 0.$$

Let $k = 2^{-1}(p + 1)$. Since $p$ is odd, $k$ is an integer and $k$ and $p$ are relatively prime. Using (2.7), define $\beta \in \Gamma$ by

$$a\beta = ka + b, \qquad (B \oplus \langle h \rangle)(\beta - 1) = 0.$$

Note that $b\tau\beta = b\tau$ since $b\tau \in pG \leq B \oplus \langle h \rangle$. One verifies that $a\beta^{-1} = 2(a - b)$, and hence

$$a\beta^{-1}\tau\beta = 2(a - b)\tau\beta = -2b\tau\beta = -2b\tau,$$

$$b\beta^{-1}\tau\beta = b\tau\beta = b\tau,$$

$$h\beta^{-1}\tau\beta = h\tau\beta = a\beta = ka + b.$$

Let $\delta = \gamma\beta^{-1}\gamma\beta = (1+\tau)(1+\beta^{-1}\tau\beta)$. Then $\delta \in N$ and $\delta = 1+\eta$, where $\eta = \tau + \beta^{-1}\tau\beta + \tau\beta^{-1}\tau\beta$. Since

$$h(\delta - 1) = h\eta = h\tau + h\beta^{-1}\tau\beta + h\tau\beta^{-1}\tau\beta$$

$$= a + (ka + b) + (-2b\tau) = (k+1)a + b - 2b\tau$$

and $b\tau \in pG$, it follows that $h(\delta - 1) \notin \langle h\rangle$. Hence $\delta \in N$ is noncentral (cf. (2.1)) and, by hypothesis, $(\delta-1)^2 = 0$. From $a, b \in G[p]$, $G[p]\tau\beta^{-1}\tau\beta \leq pG\beta^{-1}\tau\beta = pG\tau\beta = 0$, and $\tau^2 = 0$, one obtains

$$0 = h(\delta-1)^2 = [(k+1)a + b - 2b\tau]\eta$$

$$= [(k+1)a + b - 2b\tau](\tau + \beta^{-1}\tau\beta + \tau\beta^{-1}\tau\beta)$$

$$= [(k+1)a + b](\tau + \beta^{-1}\tau\beta) = (k+1)a(\tau + \beta^{-1}\tau\beta) + b(\tau + \beta^{-1}\tau\beta)$$

$$= (k+1)(-2b\tau) + b\tau + b\tau$$

$$= -2kb\tau = -2 \cdot 2^{-1}(p+1)b\tau = -b\tau.$$

Hence $b\tau = 0$, contradicting (2.8) and proving the lemma.

**Corollary 2.9.** *Let $G$ and $N$ be as in Lemma 2.5. If $N$ is noncentral then $N \cap \operatorname{fix} G[p]$ or $N \cap \operatorname{fix}(G/pG)$ is noncentral.*

3. **Proof.** Assume the situation of the Theorem and let $N$ be a noncentral normal subgroup of $\Gamma$. It was shown in [6] that cyclic normal subgroups of $\Gamma$ are central. Hence, it suffices to show that $N$ contains a noncentral elementary abelian normal $p$-subgroup of $\Gamma$. By Theorem A of [6], every noncentral normal subgroup of $\Gamma$ contains a noncentral normal subgroup of $\Gamma$ of exponent $p$. This permits the assumption

$$N^p = 1. \tag{3.1}$$

Distinguish the following cases.

*Case* 1. *$G$ is unbounded.* Then $Z\Gamma$ contains no elements of order $p$ (cf. (2.1)) and every $p$-subgroup $\neq 1$ of $\Gamma$ is noncentral. Therefore, using (3.1) and Lemma 2.4, it suffices to prove $N \cap \operatorname{fix} G[p] \neq 1$. Let, for $n \geq 1$ an integer, $\Sigma_n = \operatorname{stab} G[p^n]$. Then $\Sigma_n \leq \operatorname{fix} G[p]$ and the proof will be completed by showing $N \cap \Sigma_n \neq 1$ for some $n$. Assume, by way of contradiction, that $N \cap \Sigma_n = 1$ for all $n \geq 1$. Since $N$ and the $\Sigma_n$ are normal subgroups of $\Gamma$ this implies $N$ is contained in the centralizer $C\Sigma_n$ of $\Sigma_n$ in $\Gamma$, for all $n \geq 1$. By Lemma 2.1 of [5], $C\Sigma_n \leq Z\Gamma \cdot \operatorname{fix} G[p^n]$. Hence

$$N \leq \bigcap_{n\geq 1} C\Sigma_n \leq \bigcap_{n\geq 1} (Z\Gamma \cdot \text{fix } G[p^n]) = \Phi. \tag{3.2}$$

Using (2.1) and the fact that $G = \bigcup_{n\geq 1} G[p^n]$, one verifies that every $\phi \in \Phi$ induces the identity mapping in the lattice of all subgroups of $G$ and hence, $\phi \in Z\Gamma$. This together with (3.2) implies $N \leq Z\Gamma$ which is the desired contradiction.

*Case* 2. *G is bounded.* It has been proved in [7] that $G$ is a bounded group with two independent elements of maximal order if and only if the intersection $D\Gamma$ of all noncentral normal subgroups of $\Gamma$ is noncentral; and $D\Gamma$ is an elementary abelian $p$-group if $pG \neq 0$ [7, Theorems 2,3]. Therefore, one can assume that $G = A \oplus \langle h \rangle$ and $p^m G \neq 0 = p^m A$ for some integer $m \geq 1$. If $A$ has rank at least 2, Corollary 2.9, Lemma 2.4, and (3.1) finish the proof. Suppose that

$$G = \langle a \rangle \oplus \langle h \rangle, \qquad O(a) < O(h) = p^{m+1}. \tag{3.3}$$

If $pa \neq 0$ then $G[p] \leq pG$ and (3.1), (2.3), and (2.2) imply

$$N \leq \Psi \leq \text{stab } G[p] \simeq \text{Hom}\,(G/G[p], G[p]).$$

Hence $N$ is elementary abelian and the proof is completed.

It remains to consider the case where $a$ in (3.3) has order $p$. By (3.1) and (2.3), $N \leq \Psi$ and the elements in $\Psi$ can be identified with matrices of the form

$$\begin{bmatrix} 1 & p^m k \\ l & 1 + p^m n \end{bmatrix}$$

where $0 \leq k, l, n \leq p - 1$ are integers. Hence $\Psi$ has order $p^3$ and either $N = \Psi$ or $N$ has order $p$ or $p^2$. If $N = \Psi$ then $N$ contains noncentral elementary abelian normal $p$-subgroups of $\Gamma$, for instance stab $G[p] \leq \Psi$. If $N$ has order $p^2$ or $p$, then $N$ is commutative and, because of (3.1), $N$ is elementary abelian. (Actually, the case $|N| = p$ cannot occur since cyclic normal subgroups of $\Gamma$ are contained in $Z\Gamma$ [6, Theorem B].) The Theorem is proven.

## REFERENCES

1. R. Baer, *Primary abelian groups and their automorphisms*, Amer. J. Math. 59 (1937), 99–117.

2. J. Dieudonné, *Les déterminants sur un corps non commutatif*, Bull. Soc. Math. France 71 (1943), 27–45. MR 7, 3.

3. L. Fuchs, *Abelian groups*, Akad. Kiadó, Budapest, 1958; republished by Internat. Ser. Monographs on Pure and Appl. Math., Pergamon Press, New York, 1960. MR 21 #5672; 22 #2644.

4. J. Hausen, *On the normal structure of automorphism groups of abelian p-groups*, J. London Math. Soc. (2) 5 (1972), 409–413.

5. ———, *The automorphism group of an abelian p-group and its noncentral normal subgroups*, J. Algebra **30** (1974), 459–472.

6. ———, *The automorphism group of an abelian p-group and its normal p-subgroups*, Trans. Amer. Math. Soc. **182** (1973), 159–164.

7. ———, *Structural relations between general linear groups and automorphism groups of abelian p-groups*, Proc. London Math. Soc. (3) **28** (1974), 614–630.

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