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AN EXAMPLE INVOLVING BAIRE SPACES

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ABSTRACT. If $2^{\aleph_0} = \aleph_1$, then there is a regular, Hausdorff space Y such that (1) every subspace of Y is a Lindelöf, Baire space, (2) Y is a homogeneous space, and (3) $Y \times Y$ is of the first category.

In this note we show, using a minor modification of an argument of Oxtoby [5, pp. 163–164], that, if $2^{\aleph_0} = \aleph_1$, then there is a dense subspace Y of $(R, \mathfrak{T})$, where $\mathfrak{T}$ denotes the density topology on the real line R, which has the properties listed in the abstract. The space $(R, \mathfrak{T})$ is a completely regular, Hausdorff, Baire space with the property that

(1.1) a subset A of R is $\mathfrak{T}$-nowhere dense if and only if the Lebesgue measure of A is zero.

The density topology $\mathfrak{T}$, which consists of all Lebesgue measurable subsets of R having density 1 at each of their points, was introduced in [3] and studied in [2]; see §2 of [6] for more of its properties.

We shall denote the Euclidean topology on R by $\mathfrak{E}$. For any topological space $(X, \mathfrak{U})$, we denote by $\mathfrak{U} \times \mathfrak{U}$ the product topology on $X \times X$ induced by $\mathfrak{U}$. And, for any family $\mathfrak{F}$ of subsets of R and any subset A of R, we denote $\{F \cap A: F \in \mathfrak{F}\}$ by $\mathfrak{F} \cap A$.

Proposition. *Suppose G is an additive subgroup of R of positive Lebesgue outer measure such that G is of the first category in $(R, \mathfrak{E})$. Then $(G, \mathfrak{T} \cap G)$ is a completely regular, Hausdorff space such that* (1) *every subspace of G is a Baire space,* (2) *G is homogeneous, and* (3) *$G \times G$ is of the first category (in $G \times G$).*

Proof. It is easily seen that G is dense in $(R, \mathfrak{T})$. Next, there is a dense G_δ subset H of $(R, \mathfrak{E})$ such that $G \cap H = \varnothing$. Let $K = \{(x, y) \in R \times R: x - y \in H\}$. Then K is a G_δ subset of $(R \times R, \mathfrak{E} \times \mathfrak{E})$. And (see [4]), if A, B are Lebesgue measurable subsets of R of positive measure, then $K \cap (A \times B) \neq \varnothing$. Hence K is a dense G_δ subset of $(R \times R,$

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$\mathcal{T} \times \mathcal{T}$). Because $G \times G$ is dense in $(R \times R, \mathcal{T} \times \mathcal{T})$ and $(G \times G) \cap K = \emptyset$, $G \times G$ is of the first category.

Because every isometry of $(R, \mathcal{E})$ is a homeomorphism of $(R, \mathcal{T})$ onto itself, and G is an additive subgroup of R, $(G, \mathcal{T} \cap G)$ is homogeneous.

If $A \subset R$ and A is of the first category in $(R, \mathcal{T})$, it follows from (1.1) and the definition of $\mathcal{T}$ that $(A, \mathcal{T} \cap A)$ is discrete. It follows easily from this, that every subspace of $(R, \mathcal{T})$ is a Baire space; hence (1) holds.

In the following, we denote Lebesgue measure on R by m and the set of all rational numbers by Q. For any subset A of R and any x, y in R, we denote by $xA + y$ the set $\{xa + y: a \in A\}$.

Construction of the space Y that is described in the abstract. Suppose $2^{\aleph_0} = \aleph_1$. Let Z denote a dense G_δ subset of $(R, \mathcal{E})$ such that $0 \notin Z$ and $m(Z) = 0$. Suppose $(F_\alpha)_{\alpha<\omega_1}$ is the family of all $\mathcal{E}$-Borel sets of measure zero, and suppose $F_0 = Z$.

Let $Y_0 = \{0\}$. Suppose that, for some α, $0 < \alpha < \omega_1$, we have defined $(Y_\beta)_{\beta<\alpha}$ so that, for each β, $0 \leq \beta < \alpha$, Y_β is a countable subset of R. Let $Y^\alpha = \bigcup\{Y_\beta: \beta < \alpha\}$ and $Z^\alpha = \bigcup\{qF_\beta + y: q \in Q, \beta < \alpha, y \in Y^\alpha\}$. Because $m(Z^\alpha) = 0$, we may choose h_α in $R \sim Z^\alpha$. Let $Y_\alpha = \{qh_\alpha + y: q \in Q, y \in Y^\alpha\}$. Finally, let $Y = \bigcup\{Y_\alpha: \alpha < \omega_1\}$. It is easily seen that Y is a rational vector subspace of R. We shall now show that

$$(1.2) \qquad (Y_\alpha \sim Y^\alpha) \cap \left[\bigcup\{F_\beta: \beta < \alpha\}\right] = \emptyset \quad \text{for } 0 < \alpha < \omega_1.$$

Suppose $x \in (Y_\alpha \sim Y^\alpha) \cap [\bigcup\{F_\beta: \beta < \alpha\}]$, where $0 < \alpha < \omega_1$. Then there are $\gamma < \alpha$, q in Q, and y in Y^α such that $x \in F_\gamma$ and $x = qh_\alpha + y$. Because $x \notin Y^\alpha$, $q \neq 0$. Then $h_\alpha \in q^{-1}[F_\gamma + (-y)] \subset Z^\alpha$, contradicting the choice of h_α.

It follows from (1.2) that (a) $|Y \cap A| \leq \aleph_0$ for every subset A of R of Lebesgue measure 0, and (b) $Y \cap Z = \emptyset$. From (a) it follows that, because $|Y| > \aleph_0$, Y has positive Lebesgue outer measure. And it follows from (a), (1.1), and the fact that $(R, \mathcal{T})$ obviously satisfies the countable chain condition, that $(Y, \mathcal{T} \cap Y)$ is hereditarily Lindelöf.

Remarks. (1) Y is a totally nonmeagre space [1, p. 252] such that $Y \times Y$ is not a Baire space.

(2) It follows from [6, 2.1.2] that Blumberg's theorem does not hold for Y.

(3) To show that Y is completely regular we do not need to know that $(R, \mathcal{T})$ is completely regular; it is sufficient to know that $(R, \mathcal{T})$ is a regular, Hausdorff space. This latter statement is rather easy to prove.

(4) The space Y is not homeomorphic with the space Z constructed on p. 164 of [5] because Z is extremally disconnected and Y is not.

(5) If we replace the continuum hypothesis with the following hypothesis, then the argument used in constructing Y can be used, with only minor modifications, to construct a set G satisfying the hypothesis of the proposition.

If $\mathfrak{F}$ is a family of subsets of R of Lebesgue measure 0 such that $|\mathfrak{F}| < 2^{\aleph_0}$, then the interior Lebesgue measure of $\bigcup \mathfrak{F}$ is 0.

(6) Dr. R. M. Solovay has stated (in a letter) that (5) implies that if Martin's axiom holds, then R contains a subgroup satisfying the hypothesis of the proposition. He has stated further that it is consistent with ZF + AC that there is no subgroup of R which satisfies the hypothesis of the proposition.

(7) If G satisfies the hypothesis of the proposition, then $(G, +, \mathfrak{T} \cap G)$ is not a topological group. For, if U is an element of $\mathfrak{T}$ containing 0 such that $0 \notin \mathfrak{S}\text{-int}(\mathfrak{S}\text{-cl } U)$, then there is no V in $\mathfrak{T}$ such that $0 \in V$ and $\{x - y: x, y \in V \cap G\} \subset U \cap G$.

REFERENCES

1. N. Bourbaki, *Elements of mathematics: General topology*. Part 2, Addison-Wesley, Reading, Mass., 1966.

2. C. Goffman, C. J. Neugebauer and T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–505. MR **25** #1254.

3. C. Goffman and D. Waterman, *Approximately continuous transformations*, Proc. Amer. Math. Soc. **12** (1961), 116–121. MR **22** #11082.

4. H. Helson, *Remark on measures in almost-independent fields*, Studia Math. **10** (1948), 182–183. MR **10**, 600.

5. J. C. Oxtoby, *Cartesian products of Baire spaces*, Fund. Math. **49** (1960/61), 157–166. MR **25** #4055; **26** #1543.

6. H. E. White, Jr., *Topological spaces in which Blumberg's theorem holds*, Proc. Amer. Math. Soc. 44 (1974), 454–462.

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