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REPRESENTATION OF C^n-OPERATORS

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ABSTRACT. The operator $T_n = M + nJ$ acting in $C(I)$, where $I = [0, 1]$, $M: f(x) \longrightarrow xf(x)$, and $J: f(x) \longrightarrow \int_0^x f(t)\,dt$, is known to be of class C^n (cf. [2], [3], [4]). We show here that every real operator of class C^n in a weakly complete Banach space X has a "weak representation" as T_n.

1. Weak representation of C^n-operators. We need the following extension of [2, Lemma 2.8] to weakly complete spaces.

Lemma 1. *Let X be a weakly complete Banach space, and let T be a C^n-operator (i.e., an operator of class C^n) in X, with spectrum in I. Then the $C^n(I)$-operational calculus $T(\cdot)$ for T is uniquely representable in the form*

$$T(f) = \sum_{j=0}^{n-1} f^{(j)}(0)T^j/j! + \int_I f^{(n)}(s)F(ds), \tag{1}$$

$f \in C^n(I)$, where F is a uniformly bounded operator measure on the Borel subsets of I (F is strongly countably additive and the integral in (1) is understood in the strong operator topology). As usual, $\sum_{j=0}^{-1} a_j = 0$ for any a_j.

Proof. For $f \in C^n(I)$, we have

$$f(x) = \sum_{j=0}^{n-1} f^{(j)}(0)x^j/j! + (J^n f^{(n)})(x),$$

hence

$$T(f) = \sum_{j=0}^{n-1} f^{(j)}(0)T^j/j! + T(J^n f^{(n)}). \tag{2}$$

When f ranges in $C^n(I)$, $f^{(n)}$ ranges in the entire space $C(I)$, and it follows

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that for each y in X, the map $U_y: f^{(n)} \to T(J^n f^{(n)})y$ is a bounded linear operator from $C(I)$ into X, and it is necessarily weakly compact since X is weakly complete (cf. [1, Theorem VI.7.6]). By Theorem VI.7.3 in [1], there exists a unique vector measure m_y on the Borel sets in I, countably additive and uniformly bounded by $\|U_y\| \leq K\|y\|$, such that

$$U_y(g) = \int_I g(s) m_y(ds), \qquad g \in C(I). \tag{3}$$

For each Borel subset δ of I, write $F(\delta)y = m_y(\delta)$. Then $F(\delta) \in B(X)$, $\|F(\delta)\| \leq K$, and the Lemma follows easily from (2) and (3).

By the final remark in [5], we have the following

Corollary. *A real operator on l^1 is singular of class C^n iff it is spectral of type n.*

(See [2] for the terminology.)

Definition. The measure F in Lemma 1 will be called the *characteristic measure* of T, and the map $U: f \to \int_I f\,dF$ of $C(I)$ into $B(X)$ is then called the *characteristic operator* of T.

Definition. A *weak representation* for the C^n-operator T is a bounded operator $U: C(I) \to B(X)$ such that

$$T(f)Uh = UT_n(f)h \tag{4}$$

for all f in $C^n(I)$ and h in $C(I)$, where $T(\cdot)$ and $T_n(\cdot)$ are the operational calculi for T and T_n respectively.

Theorem 1. *Let T be a C^n-operator with spectrum in I, acting in a weakly complete Banach space. Then the characteristic operator of T is a weak representation for T.*

Proof. Since both T and T_n are of class $C^n(I)$ and polynomials are dense in $C^n(I)$, it suffices to verify that

$$T^k U = UT_n^k, \qquad k = 0, 1, 2, \cdots,$$

which in turn follows readily from the special case with $k = 1$.

By (1), we have $T(J^n h) = Uh$, $h \in C(I)$. Hence, by Lemma 1 in [3], we have for all $h \in C(I)$:

$$UT_n h = T(J^n T_n h) = T(MJ^n h) = T(xJ^n h) = T(x)T(J^n h) = TUh.$$

Remark. The reader will realize by now that in (4) one has a composition of operators on the right side and the multiplication of the operators $T(f)$ and Uh on the left side. For $n = 0$, (4) reduces to the multiplicativity property of $T(\cdot) = U$.

Theorem 2. *Let T be a bounded linear operator on the (arbitrary) Banach space X, and let n be a nonnegative integer. Suppose there exists a bounded operator $U: C(I) \to B(X)$ such that*

(i) $U1 = T^n/n!$ *and*

(ii) $TU = UT_n$.

Then T is of class $C^n(I)$ and U is its characteristic operator.

Corollary. *Let T be a bounded linear operator on the weakly complete Banach space X, and let n be a nonnegative integer. Then T is of class $C^n(I)$ iff there exists a bounded linear operator $U: C(I) \to B(X)$ satisfying* (i) *and* (ii).

Proof. Define $T(\cdot)$ on $C^n(I)$ by

$$T(f) = \sum_{j=0}^{n-1} f^{(j)}(0)T^j/j! + Uf^{(n)}, \qquad f \in C^n(I). \tag{5}$$

It suffices to verify that

$$T(x^m) = T^m \quad \text{for all } m = 0, 1, \cdots . \tag{6}$$

This is trivial for $m < n$, and it remains to show that

$$T(x^{n+k}) = T^{n+k}, \qquad k = 0, 1, \cdots . \tag{6'}$$

Since $T(x^{n+k}) = ((n+k)!/k!)Ux^k$ by (5), we must show that

$$Ux^k = k!T^{n+k}/(n+k)!, \qquad k = 0, 1, \cdots . \tag{7}$$

This is done by induction. For $k = 0$, (7) reduces to (i). Assuming (7) for k and using (ii), we obtain

$$Ux^{k+1} = \frac{k+1}{n+k+1} UT_n x^k = \frac{k+1}{n+k+1} TUx^k = \frac{(k+1)!}{(n+k+1)!} T^{n+k+1} .$$

Combining Theorems 1 and 2, we obtain the corollary.

2. **Characteristic measures.** Let F be a (weakly, hence strongly, countably additive) operator measure on the Borel sets of I, and let $U: C(I) \to B(X)$ be defined by $Uf = \int_I f\,dF$. Let M_k be the kth moment of F, $M_k = U(x^k)$,

and denote by c_δ the characteristic function of the set δ.

Theorem 3. *F is the characteristic measure of a $C^n(I)$-operator if and only if it satisfies the "characteristic identity"*

$$F(\delta)F(\epsilon) = \sum_{k=0}^{n} \binom{n}{k} U(J^k c_\delta J^{n-k} c_\epsilon) \tag{8}$$

for all Borel subsets δ, ϵ of I, and (in case $n \geq 2$) there exists an operator T such that

$$T^{n+k} = (n+k)!M_k/k!, \qquad k = 0, \cdots, n-1. \tag{9}$$

When this is the case, F is the characteristic *measure of the operator T.*

Remark. For $n = 0$, (8) reduces to the spectral measure condition $F(\delta)F(\epsilon) = F(\delta \cap \epsilon)$. For $n = 1$, the "coherence condition" (9) becomes $T = M_0$, which is just a definition of T.

Proof. *Necessity.* Suppose F is the *characteristic* measure of the $C^n(I)$-operator T. Then for all f, g in $C(I)$,

$$UfUg = T(J^n f)T(J^n g) = T(J^n f J^n g) = U((J^n f J^n g)^{(n)}),$$

and so by Leibnitz' formula

$$UfUg = \sum_{k=0}^{n} \binom{n}{k} U(J^k f J^{n-k} g) \tag{10}$$

for all f, g in $C(I)$, and hence for all bounded Borel functions f, g on I. Taking, in particular, characteristic functions, (8) follows.

For any $k \geq 0$, $x^{n+k} \in C_0^n(I) = \{f \in C^n(I);\ f^{(j)}(0) = 0,\ j = 0, \cdots, n-1\}$; hence

$$T^{n+k} = T(x^{n+k}) = U((x^{n+k})^{(n)}) = (n+k)!M_k/k!.$$

Sufficiency. The case $n = 0$ being well known, we assume $n \geq 1$. By linearity, it follows from (8) that (10) is valid for simple Borel functions, and hence for all f, g in $C(I)$. Writing $u = J^n f$, $v = J^n g$ in (10), we obtain

$$Uu^{(n)}Uv^{(n)} = U((uv)^{(n)}) \tag{11}$$

for all u, v in $J^n C(I) = C_0^n(I)$.

Let T be the operator involved in the coherence condition (9), and define $T(\cdot)$ as in (5). Again, it suffices to verify (6) for $m \geq n$. For $n \leq m \leq 2n-1$, (6) follows from (5) and (9). For $m \geq 2n$, write $m = qn + r$ uniquely with $q \geq 2$ and $r < n$, then represent x^m as a product of q $C_0^n(I)$-functions $x^m = x^{n+r}(x^n)^{q-1}$. By (5) and (11), $T(\cdot)$ is multiplicative on $C_0^n(I)$. Since (6) is valid for $m < 2n$, we obtain

$$T(x^m) = T(x^{n+r})(T(x^n))^{q-1} = T^{n+r}T^{n(q-1)} = T^m$$

as wanted.

Corollary. *F is the characteristic measure of a C^n-operator with spectrum in $R^+ = (0, \infty)$ iff (8) holds, $\sigma(M_0) \subset R^+$, and in case $n > 2$,*

$$(n!\,M_0)^{n+k} = \left[\frac{(n+k)!}{k!} M_k\right]^n, \qquad k = 1, \cdots, n-1. \tag{12}$$

Proof. The necessity part follows trivially from the corresponding part in Theorem 3.

For the converse, note that for any operator S with spectrum in R^+ and for any real α, S^α may be defined by means of the analytic operational calculus, and the composite function theorem implies that $(S^\alpha)^\beta = (S^\beta)^\alpha = S^{\alpha\beta}$ (α, β real). Define $T = (n!\,M_0)^{1/n}$. Then $\sigma(T) \subset R^+$, and $\sigma(M_k^n) \subset R^+$, $k = 0, \cdots, n-1$ (by (12)). Hence, by (12) and the composite function theorem for the analytic operational calculus,

$$T^{n+k} = [(n!\,M_0)^{n+k}]^{1/n} = \frac{(n+k)!}{k!}(M_k^n)^{1/n} = \frac{(n+k)!}{k!} M_k,$$

and the conclusion follows from Theorem 3.

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