

## ON PRODUCTS OF COUNTABLY COMPACT SPACES

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ABSTRACT. A discussion of proofs of sufficient conditions for the product of two countably compact spaces to be countably compact.

In 1953, J. Novak settled an old question of Čech by exhibiting two countably compact spaces<sup>1</sup> whose product fails to be countably compact [Nk]. He produced countably compact subsets  $N_1$  and  $N_2$  of  $\beta\mathbb{N}$  whose intersection was precisely  $\mathbb{N}$ . Thus the diagonal of  $N_1 \times N_2$  is infinite, closed and discrete, ruling out countable compactness.

Fortunately, this not too pleasant state of affairs is easily remedied by imposing an additional restriction on one of the factors. If, in addition to both factors being countably compact, one is also compact, or sequentially compact (Mrowka [M]), or first-countable (Ryll-Nardzewski [R]), or sequential (Franklin [F]), or a  $k$ -space (Noble [Nb]), then their product is countably compact. (These last results can be regarded as successive improvements since each first-countable space is sequential and each sequential space is a  $k$ -space.) It follows at once that  $N_1$  and  $N_2$  can have none of these properties and, in particular, are not  $k$ -spaces.

Recently, Burke [B, Example 1.10], apparently unaware of Noble's result, has proved that  $N_1$  is not a  $k$ -space. His argument is as follows: The intersection  $A$  of  $\mathbb{N}$  with a compact subset  $K$  of  $N_1$  is homeomorphic to the diagonal of  $\text{cl}_{N_1} A \times \text{cl}_{N_2} A$ , which is itself countably compact since the first factor is compact and second countably compact. But the diagonal is closed and discrete and, hence,  $A$  is finite. Thus  $\mathbb{N}$  meets every compact subset of  $N_1$  in a finite set but is not itself closed.

This same argument actually proves Noble's result. Suppose that  $D$  is a countable, closed, discrete subset of the product of two countably compact spaces  $C_1$  and  $C_2$ . If  $D = \{(x_i, y_i)\}$ , we may assume that the correspondence

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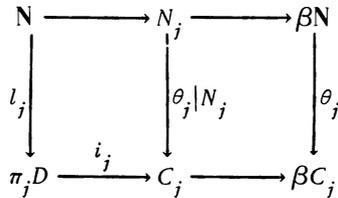
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<sup>1</sup>All spaces are presumed to be Hausdorff.

$x_i \leftrightarrow y_i$  is a bijection of the infinite sets  $\pi_1 D$  and  $\pi_2 D$ . For any compact subset  $K$  of  $C_1$ , let  $A_1 = K \cap \pi_1 D$ , and  $A_2 = \{y_i | x_i \in A_1\}$ . Then  $\{(x_i, y_i) | x_i \in A_1\}$  is contained in  $D \cap (\text{cl}_{C_1} A_1 \times \text{cl}_{C_2} A_2)$ , which must be a finite subset of the countably compact space  $\text{cl}_{C_1} A_1 \times \text{cl}_{C_2} A_2$ . Thus  $A_1$  is finite, and  $\pi_1 D$  meets each compact subset of  $C_1$  in a finite subset but is not closed. We have shown that *the projection of an infinite closed discrete subset of the product of two countably compact spaces is always  $k$ -closed but not closed.*

If one is willing to assume that the spaces in question are Tychonoff spaces, this result can be had in a much slicker fashion from Burke's observation, via the theory of perfect maps. With no loss of generality we may assume that  $\pi_1 D$  and  $\pi_2 D$  are dense in  $C_1$  and  $C_2$ , respectively, and are discrete.<sup>2</sup> Denote the indexing functions of  $D$  by  $l_j: N \rightarrow \pi_j D$ , and let  $\theta_j = \beta(i_j \circ l_j)$ , where  $i_j: \pi_j D \rightarrow C_j$  is the inclusion. Write  $N_j$  for  $\theta_j^{-1}(C_j)$ . We now have the following commutative diagram with all horizontal maps inclusions.



Since the middle arrow represents a perfect map, each  $N_j$  is countably compact. Clearly  $N_1 \times N_2$  is not countably compact, since  $C_1 \times C_2$  is not, and  $N = N_1 \cap N_2$ . Thus  $N_1$  is not a  $k$ -space, by Burke's argument. But this means that  $C_1$  is not either since  $k$ -spaces are inversely preserved by perfect maps [A, Theorem 2.5].

A smoother proof yet is available for the sequential case using results from [F - F]. If  $C_1$  and  $C_2$  are countably compact with  $C_2$  sequential, and  $S_1$  is a convergent sequence together with its limit point, the projections  $C_1 \times C_2 \times S_1 \rightarrow C_2 \times S_1 \rightarrow S_1$  are both closed and, hence, so is their composition, the projection  $C_1 \times C_2 \times S_1 \rightarrow S_1$ . Hence  $C_1 \times C_2$  is countably compact [F - F].

REFERENCES

[A] A. V. Arhangel'skiĭ, *Bicomact sets and the topology of spaces*, Dokl. Akad. Nauk SSSR 150 (1963), 9-12 = Soviet Math. Dokl. 4 (1963), 561-564. MR 27 #720; erratum, 28, p. 1246.

<sup>2</sup> I am grateful to Barbara Thomas for pointing out the need to assume discreteness.

- [B] D. K. Burke, *On  $p$ -spaces and  $w\Delta$ -spaces*, Pacific J. Math. 35 (1970), 285–296. MR 43 #3986.
- [F – F] I. Fleischer and S. P. Franklin, *On compactness and projections*, Proc. Internat. Sympos. Ext. Theory of Topological Structures and Appl., VEB Deutscher Verlag, Berlin, 1969, pp. 77–79.
- [F] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. 57 (1965), 107–115. MR 31 #5184.
- [M] S. Mrowka, *Compactness and product spaces*, Colloq. Math. 7 (1959), 19–23.
- [Nb] N. Noble, *Countably compact and pseudocompact products*, Czechoslovak Math. J. 19 (94) (1969), 390–397. MR 40 #1968.
- [Nk] J. Novak, *On the cartesian product of two compact spaces*, Fund. Math. 40 (1953), 106–112.
- [R] C. Ryll-Nardzewski, *A remark on the cartesian product of two compact spaces*, Bull. Acad. Polon. Sci. Cl. III 2 (1934), 265–266.

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