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RESTRICTED CENTERS IN $C(\Omega)$

PHILIP W. SMITH AND J. D. WARD[1]

ABSTRACT. The concept of restricted center is a natural generalization of the notion of Chebyshev center. We prove a necessary and sufficient condition for a bounded subset A of $C(\Omega)$, Ω paracompact, to have a restricted center with respect to B, another subset of $C(\Omega)$. This theorem is then applied to subspaces of finite codimension in $C(I)$, I a compact interval.

1. Introduction. This paper is concerned with Chebyshev centers in the space $C(\Omega)$, where Ω is a compact or paracompact Hausdorff topological space, and $C(\Omega)$ denotes the set of bounded real-valued continuous functions on Ω. Of course, $C(\Omega)$ is a Banach space when supplied with the supremum norm, $\|\cdot\|$. If I is a compact interval in the real line and B is a bounded subset of $C(I)$, then Kadeč and Zamyatin [4] have shown that the problem

$$\inf_{x \in C(I)} \sup_{y \in B} \|x - y\| \equiv R_{C(I)}(B) \tag{1.1}$$

has a solution. That is, there is an $x_* \in C(I)$ so that $\sup_{y \in B}\|x_* - y\| = R_{C(I)}(B)$. The solutions to (1.1) are called Chebyshev centers, and the number $R_{C(I)}(B)$ is called the Chebyshev radius of B. We will consider a slightly more general problem than (1.1); e.g., for any $G \subset C(\Omega)$ we consider

$$\inf_{x \in G} \sup_{y \in B} \|x - y\| \equiv R_G(B). \tag{1.2}$$

Solutions to (1.2), if they exist, will be called restricted centers of B with respect to G. The set of solutions will be denoted by $E_G(B)$ and the restricted radius is $R_G(B)$.

In § 2 we will derive a necessary and sufficient condition for (1.2) to have a solution. The main tool of this section is Michael's theorem on continuous selections [5], and a generalization to paracompact spaces due

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to Holmes [2] of the Kadeć and Zamyatin result [4] mentioned above.

In § 3 we will make repeated applications of the results in § 2. In this section we consider only hyperplanes of finite codimension. In particular, we answer a question of Holmes [3] concerning the existence of a Banach space in which all compact sets have Chebyshev centers, but not all bounded sets have Chebyshev centers. The answer is that such a Banach space exists and in fact it is a Chebyshev hyperplane of $C(I)$ with codimension one.

It is the feeling of the authors that the study of Chebyshev centers in general Banach spaces will be greatly enhanced by a detailed study of the center problem in $C[0, 1]$. This is due in part to the fact that $C[0, 1]$ is universal for separable Banach spaces. For results on finite dimensional subspaces of $C(I)$ see [6]; for results on closed subalgebras of $C(\Omega)$ see [8]. Of the many references on the Chebyshev center problem in general Banach spaces see, for example, Garkavi's fundamental paper [1] and the very readable presentation in [2].

2. **Existence of restricted centers.** Let B be a bounded nonempty subset of $C(\Omega)$. Define

$$(2.1)\qquad \begin{aligned} m(t) &= \inf\{x(t)\colon x \in B\}, & M(t) &= \sup\{x(t)\colon x \in B\}, \\ n(t) &= \liminf\{m(s)\colon s \to t\}, & N(t) &= \limsup\{M(s)\colon s \to t\}. \end{aligned}$$

It is easy to see that N and m are u.s.c., upper semicontinuous, on Ω and that n and M are l.s.c., lower semicontinuous, on Ω. Setting $r = \frac{1}{2}\sup_{t\in\Omega}(N(t) - n(t))$, the following theorem was obtained [2], [4].

Theorem 2.1. *If B is any bounded subset of $C(\Omega)$, Ω paracompact, then $E_{C(\Omega)}(B) \neq \emptyset$ and*

$$(2.2)\qquad E_{C(\Omega)}(B) = \{x \in C(\Omega)\colon N - r \le x \le n + r\}.$$

Furthermore, $r = R_{C(\Omega)}(B)$.

A very accessible proof of this theorem may be found in [2]. We now state a general, necessary and sufficient condition for the existence of restricted centers.

Theorem 2.2. *Let Ω be paracompact, B a bounded set in $C(\Omega)$, and $G \subset C(\Omega)$. A necessary and sufficient condition for $E_G(B)$ to be nonempty is*

$$(2.3)\qquad R = \inf\{\|x - y\|\colon x \in G,\ y \in E_{C(\Omega)}(B)\} = \|x_* - y_*\|,$$

where $x_* \in G$ *and* $y_* \in E_{C(\Omega)}(B)$. *Furthermore* $R_G(B) = R_{C(\Omega)}(B) + R$.

Proof. Before proving Theorem 2.2, we first establish

$$R_G(B) \geq R_{C(\Omega)}(B) + R. \tag{2.4}$$

Let $S_k = R_G(B) - R_{C(\Omega)}(B) + 1/k$, $k = 1, 2, \cdots$. Let $g_k \in G$ satisfy

$$\sup_{b \in B} \|g_k - b\| \leq R_G(B) + 1/k. \tag{2.5}$$

Then using the notation in (2.1), we have for all $t \in \Omega$,

$$N(t) - R_{C(\Omega)}(B) - S_k \leq g_k(t) \leq n(t) + R_{C(\Omega)}(B) + S_k. \tag{2.6}$$

Now consider the set-valued map

$$\Phi_k(t) = [g_k(t) - S_k,\ g_k(t) + S_k] \cap [N(t) - R_{C(\Omega)}(B),\ n(t) + R_{C(\Omega)}(B)]. \tag{2.7}$$

It is easy to see that $\Phi_k(t)$ is closed, convex and bounded for all $t \in \Omega$. The inequality in (2.6) guarantees that $\Phi_k(t)$ is nonempty for all $t \in \Omega$.

To show that Φ_k is l.s.c., let $\mathcal{O} = (\alpha, \beta)$ and suppose that $\Phi_k(t_0) \cap \mathcal{O} \neq \emptyset$. Without loss of generality, assume $N(t_0) - R_{C(\Omega)}(B) = n(t_0) + R_{C(\Omega)}(B) = p$. Then we may choose an $\alpha_1 \in \mathcal{O}$ so that $\alpha_1 < p$. Since n is l.s.c. there is a neighborhood of t_0, Q_1, so that

$$n(t) + R_{C(\Omega)}(B) > \alpha_1, \qquad t \in Q_1. \tag{2.8}$$

Similarly there is an $\alpha_2 > p$ and a neighborhood of t_0, Q_2, so that

$$N(t) - R_{C(\Omega)}(B) < \alpha_2, \qquad t \in Q_2. \tag{2.9}$$

Since g_k is continuous there is a neighborhood Q_3 of t_0 so that

$$[g_k(t) - S_k,\ g_k(t) + S_k] \cap [\alpha_1, \alpha_2] \neq \emptyset, \qquad t \in Q_3. \tag{2.10}$$

Thus, $Q = Q_1 \cap Q_2 \cap Q_3$ is a neighborhood of t_0 satisfying

$$\Phi_k(t) \cap \mathcal{O} \neq \emptyset, \qquad t \in Q. \tag{2.11}$$

Therefore, Φ_k is l.s.c.

Appealing to Michael's theorem [5], we find that there is a $y_k \in E_{C(\Omega)}(B)$ satisfying

$$\|g_k - y_k\| \leq S_k, \tag{2.12}$$

and hence $S_k \geq R$ by (2.3) for all k. Thus (2.4) must hold.

We now prove Theorem 2.2. Condition (2.3) is seen to be sufficient since

$$\sup_{b \in B} \|b - x_*\| \leq \sup_{b \in B} \{\|x_* - y_*\| + \|y_* - b\|\} \leq R + R_{C(\Omega)}(B) \leq R_G(B) \tag{2.13}$$

implies that $x_* \in E_G(B)$. Conversely, if $x_* \in E_G(B)$, an argument similar to the one above will show that

$$\Phi(t) = [x_*(t) - R,\ x_*(t) + R] \cap [N(t) - R_{C(\Omega)}(B),\ n(t) + R_{C(\Omega)}(B)] \tag{2.14}$$

is l.s.c. and, hence, there is a continuous selection $y_* \in E_{C(\Omega)}(B)$ so that (2.3) holds. This completes the proof of the theorem since it follows easily that $R_G(B) \leq R + R_{C(\Omega)}$.

3. **Application to subspaces of finite codimension.** A set A in a Banach space X is called proximinal if the problem $\inf\{\|x - y\|: y \in A\}$ has a solution for all $x \in X$. Considering problem (1.2), with B a singleton, it is easy to see that a set $A \subset C(\Omega)$ must be proximinal if $E_A(B) \neq \emptyset$ for all bounded subsets B of $C(\Omega)$. Thus, the importance of the next theorem lies in the fact that we are not considering all bounded subsets of $C(\Omega)$.

Theorem 3.1. *A necessary condition for a hyperplane V of finite codimension in $C[a, b]$ to satisfy*

$$E_V(B) \neq \emptyset \quad \textit{for all bounded } B \subset V \tag{3.1}$$

is that V be proximinal.

Proof. We may assume that $V = \bigcap_{i=1}^{n} \mu_i^{-1}(0)$, where the $\mu_i \in C^*[a, b]$ are linearly independent. We will identify μ_i with its corresponding regular borel measure which represents μ_i. Let $s \in C[a, b] \backslash V$ and suppose

$$\inf_{x \in V} \|x - s\| = r > 0. \tag{3.2}$$

The set $B = \{x \in V: \|x - s\| < 2r\}$ is a bounded nonempty subset of V. We will show that $E_{C[a,b]}(B) = s$. Once we have done this, the result will follow immediately since the double inf of Theorem 2.2 becomes a best approximation statement about V $(= G)$. To show that $E_{C[a,b]}(B) = s$, we will use the characterization in Theorem 2.1.

Lemma 3.1. *With B as above we have $n(t) = s(t) - 2r$ and $N(t) = s(t) + 2r$.*

Proof. Surely $n(t) \geq s(t) - 2r$ and $N(t) \leq s(t) + 2r$. Suppose for some $t \in [a, b]$

$$N(t) = s(t) + 2r - 4\epsilon \tag{3.3}$$

for some $\epsilon > 0$ and $\epsilon < r$. For any $d > 0$, let $Q_d = (t - d, t + d)$. There is an open set $Q_d^1 \subset [a, b]$ satisfying

$$\text{(3.4)} \qquad \begin{aligned} &\text{(a) } Q_d \setminus \overline{Q_d^1} \neq \varnothing \quad \text{and} \\ &\text{(b) } \{\mu_i|_{Q_d^1}\}_{i=1}^n \text{ are linearly independent.} \end{aligned}$$

Here the notation $\mu_i|_Q$ means to view μ_i as a measure on Q. We note that such a Q_d^1 must exist, since otherwise we could take in Q_d a nested sequence $Q^i \subset Q^{i-1}$ of open subsets with closure in Q_d and common empty intersection such that

$$\text{(3.5)} \qquad \mu_j\left(\bigcap_{i=1}^{\infty} \overline{Q^i}\right) = 0, \qquad j = 1, \cdots, n.$$

Then let $A^i = (\overline{Q^i})^c$. If $\{\mu_j|_{A^i}\}_{j=1}^n$ are linearly dependent for all i, it follows that the $\{\mu_i\}_{i=1}^n$ are linearly dependent on $[a, b]$, a contradiction. Let Q^k be as in (3.5) with the further stipulation that

$$\text{(3.6)} \qquad Q^k \subset Q_d \setminus Q_d^1, \qquad k = 1, 2, \cdots.$$

Thus, the total variation, $|\mu_i|(Q^k)$, goes to zero uniformly in i as $k \to \infty$ by (3.5). We choose a $v \in V$ so that

$$\text{(3.7)} \qquad \|v - s\| \leq r + \epsilon.$$

Choose $g_k \in C_c(Q^k)$, i.e. $\operatorname{supp}(g_k) \subset Q^k$, so that

$$\text{(3.8)} \qquad \begin{aligned} &0 \leq g_k(\tau) \leq s(\tau) + 2r - \epsilon - v(\tau) \quad \text{for all } \tau \in Q^k, \text{ and} \\ &g_k(\tau_0) \geq s(\tau_0) + 2r - 2\epsilon - v(\tau_0) \quad \text{for some } \tau_0 \in Q^k. \end{aligned}$$

We note that $\|g_k\| \leq 3r$, and, hence, from the remark following (3.6), $\mu_i(g_k) \to 0$ uniformly in i as $k \to \infty$. Now consider the set

$$\text{(3.9)} \qquad L_k = \{x \in C_c(Q_d^1) : \mu_i(x) = \mu_i(x) = \mu_i(g_k),\ i = 1, \cdots, n\}.$$

For k large enough there is a $z_k \in L_k$ so that $\|z_k\| \leq \epsilon$. Using this k we see by (3.7), (3.8), and (3.9) that

$$\text{(3.10)} \qquad \begin{aligned} &(g_k - z_k) \in V, \quad \|v + g_k - z_k - s\| \leq 2r, \quad \text{and} \\ &v(\tau_0) + g_k(\tau_0) - z_k(\tau_0) \geq s(\tau_0) + 2r - 2\epsilon, \qquad \tau_0 \in Q^k. \end{aligned}$$

Hence $(v + g_k - z_k) \in V$ and its value at τ_0 is larger than $s(\tau_0) + 2r - 2\epsilon$. Since τ_0 is within d of t and d was an arbitrary positive number, it follows that $N(t) \geq s(t) + 2r - 2\epsilon$, which contradicts our assumptions on N and, therefore, $N(t) = s(t) + 2r$. Similarly, $n(t) = s(t) - 2r$ which proves the Lemma. Appealing to Theorem 2.1 we see that $E_{C[a,b]}(B) = s$. Recalling the remarks at the beginning of the proof completes the proof of Theorem 3.1.

We remark at this point that there is a characterization of proximinal finite codimensional hyperplanes of $C[a, b]$ due to Garkavi which may be found in [7]. The fact that proximinality is not sufficient (even for codimension one) to guarantee the existence of centers is contained in

Proposition 3.1. *Let* $V = \{x \in C[0, 2]: \int_0^2 x(t)\,dt = 0\}$. *Then there is a bounded set* $B \subset V$ *so that* $E_V(B) = \varnothing$.

Proof. Let

(3.10) $B = \{x \in V: -2 \leq x(t) \leq 4 \text{ for } t \in [0, 1] \text{ and } 0 \leq x(t) \leq 3 \text{ for } t \in [1, 2]\}$.

From (3.10) we can calculate n and N as in (2.1). Theorem 2.1 shows that $R_{C[0,2]}(B) = 3$ and $E_{C[0,2]}(B) = \{x \in C[0, 2]: N - 3 \leq x \leq n + 3\}$. Note that $N - 3$ is nonnegative. Theorem 2.2 shows that $E_V(B) \neq \varnothing$ if and only if (2.3) is met. It is well known that the distance of a function $g \in C[0, 2]$ from V is given by $\frac{1}{2}|\int_0^2 g(t)\, dt|$, and the best approximant from V is unique and is given by $g - \frac{1}{2}\int_0^2 g(t)\,dt$. Thus, condition (2.3) is met if and only if we can solve

$$r = \inf\left\{\left|\int_0^2 x(t)\,dt\right| : x \in E_{C[0,2]}(B)\right\}. \tag{3.11}$$

But (3.11) clearly has no solution since the lower bound $r = \int_0^2 (N(t) - 3)dt$, and N has a jump discontinuity at $t = 1$. This completes the proof of Proposition 3.1.

Remark. If we consider V as a Banach space in its own right then we have shown that the center problem does not have a solution for all bounded sets. However, as will be shown in the next proposition, $E_V(A) \neq \varnothing$ for all compact subsets A. This means that V is a Banach space which has centers for all compact subsets, but not for all bounded sets. This observation answers affirmatively a question of Holmes [3].

Proposition 3.2. *Let* Ω *be paracompact and* $H = \sigma^{-1}(0)$, $\|\sigma\| = 1$, *and* $\sigma \in C^*(\Omega)$. *Then* $E_H(B) \neq \varnothing$ *for all compact subsets* B *of* $C(\Omega)$ *if and only if* σ *attains its norm on the unit ball of* $C(\Omega)$.

Proof. The "only if" portion of this proposition is trivial, since if σ does not attain its norm on the unit ball then H is antiproximinal. We know that

$$(3.12)\quad \inf\{\|x-y\|: x \in H,\ y \in E_{C(\Omega)}(B)\} = \inf\{|\sigma(y)|: y \in E_{C(\Omega)}(B)\}.$$

Since H is proximinal, recall σ attains its norm; it follows that the right side of (3.12) attains its norm if and only if the left side attains. Now B is compact so that n and N defined in (2.1) are continuous. Viewing σ as a measure, we set

$$(3.13)\qquad N_s = \operatorname{supp}(\sigma^-) \quad \text{and} \quad P_s = \operatorname{supp}(\sigma^+)$$

when $\sigma = \sigma^+ - \sigma^-$ is the Jordan decomposition of σ. Following the argument in [9] we construct w_1 and w_2 in $E_{C(\Omega)}(B)$ so that for all $z \in E_{C(\Omega)}(B)$

$$(3.14)\qquad \begin{aligned} w_1(t) \le z(t) \le w_2(t), &\qquad t \in P_s, \\ w_1(t) \ge z(t) \ge w_2(t), &\qquad t \in N_s. \end{aligned}$$

It follows that $\sigma(w_2 - w_1) \ge 0$, and for any $z \in E_{C(\Omega)}(B)$

$$(3.15)\qquad \sigma(w_2) \ge \sigma(z) \ge \sigma(w_1).$$

The inf in (3.12) is attained at either w_1 or w_2 or $H \cap E_{C(\Omega)}(B) \ne \emptyset$, in which case the inf is zero and is attained. Applying Theorem 2.2 completes the proof.

If we consider all bounded subsets of $C[a, b]$, then a necessary and sufficient condition for a finite codimensional hyperplane to have restricted centers is contained in

Proposition 3.3. *Let Ω be compact and $V = \bigcap_{i=1}^{k} \sigma_i^{-1}(0)$, where $\sigma_i \in C^*(\Omega)$. Then $E_V(B) \ne \emptyset$ for every bounded subset of B of $C(\Omega)$ if and only if the supports of the σ_i are finite.*

This theorem is due to V. N. Zamyatin [10]. It is not true if we weaken "Ω compact" to "Ω paracompact".

4. **Remarks.** Let $\sigma \in C^*[a, b]$, $\|\sigma\| = 1$, and suppose that σ does not attain its norm on the unit sphere of $C[a, b]$. Then $H = \sigma^{-1}(0)$ is antiproximinal, and by Theorem 3.1 we can expect $E_H(B) = \emptyset$ for some bounded subsets of H. Let B be a bounded subset so that $E_{C(\Omega)}(B)$ is bounded away from H. Kadeć and Zamyatin [4] have shown that the map $E_{C(\Omega)}(\cdot)$ defined by

$$(4.1)\qquad B \mapsto E_{C(\Omega)}(B), \qquad B \text{ bounded},$$

is continuous in the sense of the Hausdorff metric. Therefore, whenever $E_{C(\Omega)}(B)$ is bounded away from H, then for all sets $\tilde{B}$ near enough to B

in the Hausdorff metric, we have $E_{C(\Omega)}(\tilde{B})$ is bounded away from H and, hence, $E_H(\tilde{B}) = \varnothing$. This indicates that there are many bounded sets $B \subset H$ so that $E_H(B) = \varnothing$.

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843*

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47906

* Current address of both authors.