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$\mathfrak{F}$-PROJECTORS OF FINITE SOLVABLE GROUPS

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ABSTRACT. $\mathfrak{F}$ denotes a class of finite solvable groups closed under the taking of epimorphic images. For any finite solvable group G we construct $\mathfrak{F}$-projectors and prove that they are a single conjugacy class of subgroups of G.

0. **Introduction.** The concept of a formation was introduced by Gäschutz [3] and showed that in finite solvable groups the Sylow subgroups and Carter subgroups are examples of more general classes called $\mathfrak{F}$-projectors. If $\mathfrak{F}$ is a saturated formation then every finite solvable group has a single conjugacy class of $\mathfrak{F}$-projectors. Some results of formation theory have been extended to less restricted classes; see [2], [5], and [7]. In [8] Wielandt localized the concept of a formation by studying $Q(G)$, the factors of a single group G, and showed that only the properties of the collection $\mathfrak{F} \cap Q(G)$ are relevant to the existence and conjugacy of $\mathfrak{F}$-projectors of G. Results analogous to this have been obtained for $\mathfrak{F}$-normalizers by Prentice [6]. In this note we follow the approach of studying the factors of a single group G and construct $\mathfrak{F}$-projectors for any class of factors closed under the taking of epimorphic images within G. The subgroups we obtain coincide with those of Schunk if $\mathfrak{F}$ is a saturated homomorph.

1. **$\mathfrak{F}$-projectors.** All groups considered are finite and solvable.

Let G be a group. We define $Q(G) = \{H/H_0 \mid H_0 \trianglelefteq H \leq G\}$, the set of factors of G. Let $\mathfrak{F}$ be any subset of $Q(G)$.

Definition 1.1. (i) A subgroup E of G has the $\mathfrak{F}$-covering property in G if whenever $E \leq F \leq G$ and $F/F_0 \in \mathfrak{F}$ then $EF_0 = F$.

(ii) A subgroup E of G with the $\mathfrak{F}$-covering property in G is an $\mathfrak{F}$-projector of G if no proper subgroup of E has the $\mathfrak{F}$-covering property in E.

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If $\mathfrak{F}^*$ is a saturated formation and $\mathfrak{F} = \mathfrak{F}^* \cap Q(G)$, the above definition coincides with the usual one.

We will determine conditions on $\mathfrak{F}$ that insure that $\mathfrak{F}$-projectors of G exist and are a single conjugacy class of subgroups of G.

Definition 1.2. $\mathfrak{F}$ is a homomorph if

(1) $A_0 \trianglelefteq A$, $B_0 \trianglelefteq B$, $A_0B = A$ and $A_0 \cap B = B_0$, then $A/A_0 \in \mathfrak{F}$ if and only if $B/B_0 \in \mathfrak{F}$; and

(2) $R/R_0 \in \mathfrak{F}$ and $R_0 \leq R_1 \trianglelefteq R$, then $R/R_1 \in \mathfrak{F}$.

Definition 1.3. $\mathfrak{F}$ is normal if and only if whenever $R/S \in \mathfrak{F}$ and $g \in G$ then $R^g/S^g \in \mathfrak{F}$.

If $\mathfrak{F}$ is closed with respect to the taking of epimorphic images within G, then Definitions 1.2 and 1.3 are satisfied.

In the rest of the paper, $\mathfrak{F}$ will denote a normal homomorph. If $H \leq G$, then $\mathfrak{F} \cap Q(H)$ is a normal homomorph of H. If $N \trianglelefteq G$ and we identify the factor $(L/N)/(L_0/N)$ of G/N with the factor L/L_0 of G, then the set $\mathfrak{F} \cap Q(G/N)$ is a normal homomorph of G/N. We will refer to $\mathfrak{F} \cap Q(H)$-projectors of H and $F \cap Q(G/N)$-projectors of G/N as $\mathfrak{F}$-projectors of G and G/N, respectively.

Proposition 1.4. *Let E be an $\mathfrak{F}$-projector of G.*

(1) *If $E \leq H \leq G$ then E is an $\mathfrak{F}$-projector of H.*

(2) *If $N \trianglelefteq G$ then EN/N is an $\mathfrak{F}$-projector of G/N.*

Proof. (1) is clear from the definition.

(2) Let $EN/N \leq F/N$ and suppose $N \leq F_0 \trianglelefteq F$ and $F/F_0 \in \mathfrak{F}$. Then since $E \leq F$ and E is an $\mathfrak{F}$-projector of G, $EF_0 = F$, hence $(EN/N) \cdot (F_0/N) = F/N$. Now suppose E_1/N has the $\mathfrak{F}$-covering property in EN/N. Suppose $E/E_0 \in \mathfrak{F}$. Then $EN/E_0N \in \mathfrak{F}$ so $E_1(E_0N) = EN$. Hence since $E_1 \geq N$, $E_1E_0 = EN$ so $E = (E \cap E_1)E_0$ by Dedekind's lemma. Thus $E \cap E_1$ has the $\mathfrak{F}$-covering property in E and so $E \cap E_1 = E$. Therefore $EN = E_1$. Thus EN/N is an $\mathfrak{F}$-projector of G/N.

Lemma 1.5. *Let E/N be an $\mathfrak{F}$-projector of G/N and let D be an $\mathfrak{F}$-projector of E. Then D is an $\mathfrak{F}$-projector of G.*

Proof. By (2) of Proposition 1.4, DN/N is an $\mathfrak{F}$-projector of E/N. Therefore $DN = E$ since the only subgroup of E/N which has the $\mathfrak{F}$-covering property in E/N is E/N itself. Now suppose $D \leq F$ and $F/F_0 \in \mathfrak{F}$. Then $E/N = DN/N \leq FN/N$. Hence $E(F_0N) = FN$, i.e., $EF_0 = FN$. Thus $F = (E \cap F)F_0$. Thus $(E \cap F)/(E \cap F_0) \in \mathfrak{F}$ and $D \leq E \cap F$. Therefore $D(E \cap F_0)$

$= E \cap F$, so $D(E \cap F_0)F_0 = F$, i.e., $DF_0 = F$. Thus D is an $\mathfrak{F}$-projector of G.

Theorem 1.6. *If $\mathfrak{F}$ is a normal homomorph of G, then G has an $\mathfrak{F}$-projector and the $\mathfrak{F}$-projectors of G are a single conjugacy class of subgroups of G.*

Proof. *Existence.* We proceed by induction on $|G|$. Let N be a minimal normal subgroup of G. By induction G/N has an $\mathfrak{F}$-projector E/N. If $E < G$ then E has an $\mathfrak{F}$-projector D also by induction. Then D is an $\mathfrak{F}$-projector of G by Lemma 1.5. Thus we may assume that G/N is its own $\mathfrak{F}$-projector. Now if G is its own $\mathfrak{F}$-projector there is nothing to show, so assume it is not. Then there must exist a subgroup $E < G$ with the $\mathfrak{F}$-covering property in G. Suppose that $E_1 \leq E$ has the $\mathfrak{F}$-covering property in E. Since EN/N has the $\mathfrak{F}$-covering property in G/N, $EN/N = G/N$ and so $E \cap N = 1$. But E_1N/N has the $\mathfrak{F}$-covering property in EN/N so we must also have $E_1N = G$. Therefore since $E_1 \leq E$, $E_1 = E$. Thus E is an $\mathfrak{F}$-projector of G.

Conjugacy. Clearly, if E is an $\mathfrak{F}$-projector of G so is any conjugate of E, thus we only need to show that any two $\mathfrak{F}$-projectors of G are conjugate. We proceed by induction on $|G|$. If G is its own $\mathfrak{F}$-projector there is nothing to show, so assume E, F are $\mathfrak{F}$-projectors of G and $E, F < G$. Let N be a minimal normal subgroup of G. Then by (2) of Proposition 1.4, EN/N and FN/N are $\mathfrak{F}$-projectors of G/N so, by induction, $EN = F^gN$ for some $g \in G$. Since F^g is also an $\mathfrak{F}$-projector, we may assume $EN = FN$. Now if $EN < G$ we may apply induction in EN to conclude that E and F are conjugate because of (1) of Proposition 1.4. Thus we may assume $EN = FN = G$ for every minimal normal subgroup N of G. Thus E and F are maximal subgroups of G, and $\operatorname{core}_G E = \operatorname{core}_G F = 1$ so E and F are conjugate by Ore's theorem [4, p. 165].

2. $\mathfrak{F}$-crucial chains.

In this section the $\mathfrak{F}$-projectors of G are characterized by means of the maximal chains joining them to G.

Definition 2.1. A complemented chief factor H/K of G is $\mathfrak{F}$-crucial if G/H is its own $\mathfrak{F}$-projector but G/K is not. A complement M to H/K is an $\mathfrak{F}$-crucial maximal subgroup.

Lemma 2.2. *If M is an $\mathfrak{F}$-crucial maximal subgroup of G which complements the $\mathfrak{F}$-crucial chief factor H/K, then M/K is an $\mathfrak{F}$-projector of G/K.*

Proof. Without loss of generality, we may assume $K = 1$ so H is a minimal normal subgroup of G. If M does not have the $\mathfrak{F}$-covering property in G, then $ML \neq G$ for some $L \trianglelefteq G$ such that $G/L \in \mathfrak{F}$. Since M is a maximal subgroup of G, it follows that $L \leq M$. Let E be an $\mathfrak{F}$-projector of G. Since G/H is its own $\mathfrak{F}$-projector, $EH = G$. Also $EL = G$ since $G/L \in \mathfrak{F}$. Thus $M = M \cap EL = (M \cap E)L$. Let $N_1 = E \cap LH$. Then $N_1 \trianglelefteq E$ and $H \leq C_G(N_1)$ since $LH = L \times H = LN_1$. Thus $N_1 \trianglelefteq G$. Now $N_1 \nleq M$ since $L \leq M$ but $LN_1 = LH \nleq M$. Thus $MN_1 = G$. Now suppose $M/M_0 \in \mathfrak{F}$. Then $G/M_0N_1 \in \mathfrak{F}$, hence $EM_0N_1 = G$, so $EM_0 = G$. Therefore $M = (M \cap E)M_0$. Therefore $M \cap E$ has the $\mathfrak{F}$-covering property in M. However $M \cong G/H$ is its own $\mathfrak{F}$-projector. Therefore, $M \leq E$ and $E < G$ since G is not its own $\mathfrak{F}$-projector. Hence $M = E$ contrary to assumption. Thus M is an $\mathfrak{F}$-projector of G.

Lemma 2.3. *G is its own $\mathfrak{F}$-projector if and only if G possesses no $\mathfrak{F}$-crucial maximal subgroups.*

Proof. If G has an $\mathfrak{F}$-crucial maximal subgroup, then G is not its own $\mathfrak{F}$-projector by Lemma 2.2. Let E be an $\mathfrak{F}$-projector of G and suppose $E < G$. Then there is a chief factor H/K of G such that $EH = G$ but $EK < G$. Then by (2) of Proposition 1.4, $G/H = EH/H$ is its own $\mathfrak{F}$-projector but $G/K > EK/K$ is not. Thus EK is an $\mathfrak{F}$-crucial maximal subgroup of G.

Definition 2.4. A sequence $G = M_0 > M_1 > \cdots > M_n$ is an $\mathfrak{F}$-crucial maximal chain if M_i is an $\mathfrak{F}$-crucial maximal subgroup of M_{i-1} for $i \geq 1$ and M_n has no $\mathfrak{F}$-crucial maximal subgroups.

Theorem 2.5. (i) *If $G = M_0 > \cdots > M_n$ is an $\mathfrak{F}$-crucial maximal chain, then M_n is an $\mathfrak{F}$-projector of G.*

(ii) *If E is an $\mathfrak{F}$-projector of G, then there is an $\mathfrak{F}$-crucial maximal chain $G = M_0 > M_1 > \cdots > M_n$ such that $E = M_n$.*

Proof. (i) We proceed by induction on n. If $n = 0$ then G has no $\mathfrak{F}$-crucial maximal subgroups so G is its own $\mathfrak{F}$-projector and there is nothing to prove. If $n > 0$ let M_1 complement the $\mathfrak{F}$-crucial chief factor H/K. Then an $\mathfrak{F}$-projector of M_1 is an $\mathfrak{F}$-projector of G by Lemma 1.5. However, M_n is an $\mathfrak{F}$-projector of M_1 by induction so we are done.

(ii) Since $\mathfrak{F}$ is normal and M is an $\mathfrak{F}$-crucial maximal subgroup of M_{i-1}, if $g \in G$, then M_i^g is an $\mathfrak{F}$-crucial maximal subgroup of M_{i-1}^g. Since any two $\mathfrak{F}$-projectors are conjugate, (ii) now follows from (i).

3. **Examples.** In this section $\mathfrak{X}$ will denote a class of groups closed under the taking of epimorphic images. If G is a group then G has $\mathfrak{X} \cap Q(G)$-

projectors which we will refer to as $\mathfrak{X}$-projectors. In general, the $\mathfrak{X}$-projectors of G are not simply the $\mathfrak{Y}$-projectors of G for some saturated formation $\mathfrak{Y}$ as the next theorem shows.

Theorem 3.1. *Let $\mathfrak{Y}$ be a saturated formation such that $\mathfrak{X} \leq \mathfrak{Y}$. For every group G, the $\mathfrak{X}$-projectors of G coincide with the $\mathfrak{Y}$-projectors of G if and only if $\mathfrak{Y} = \phi R_0 \mathfrak{X}$.*

Proof. (i) Suppose that for every group G the $\mathfrak{X}$-projectors of G coincide with the $\mathfrak{Y}$-projectors of G. Since $\mathfrak{Y}$ is saturated, $\phi R_0 \mathfrak{X} \leq \mathfrak{Y}$. Let $G \in \mathfrak{Y}$. G is it own $\mathfrak{X}$-projector by assumption, hence G has no $\mathfrak{X}$-crucial maximal subgroups. Therefore $G/\mathrm{core}_G M$ is its own $\mathfrak{X}$-projector for every maximal subgroup M of G. Since $M/\mathrm{core}_G M$ does not have the $\mathfrak{X}$-covering property in G, there exists $N/\mathrm{core}_G M \lhd G/\mathrm{core}_G M$ such that $G/N \in \mathfrak{X}$ but $MN \neq G$. However, $G/\mathrm{core}_G M$ is monolithic so $N = \mathrm{core}_G M$. Thus $G/\mathrm{core}_G M \in \mathfrak{X}$ for every maximal subgroup M of G. Let $C = \bigcap \{\mathrm{core}_G M \mid M$ is a maximal subgroup of $G\}$. Then $G/C \in R_0 \mathfrak{X}$ and $C = \phi(G)$, so $G \in \phi R_0 \mathfrak{X}$.

(ii) Suppose now that $\mathfrak{Y} = \phi R_0 \mathfrak{X}$ is a saturated formation. Let G be a group; we prove by induction on $|G|$ that the $\mathfrak{X}$-projectors of G coincide with the $\mathfrak{Y}$-projectors of G. If G is its own $\mathfrak{X}$-projector, then G has no $\mathfrak{X}$-crucial maximal subgroups, and the proof of (i) shows that $G \in \phi R_0 \mathfrak{X} = \mathfrak{Y}$, so G is its own $\mathfrak{Y}$-projector. If G is not its own $\mathfrak{X}$-projector then G possesses an $\mathfrak{X}$-crucial maximal subgroup M which complements an $\mathfrak{X}$-crucial chief factor H/K. Then G/H is its own $\mathfrak{X}$-projector, so $G/H \in \mathfrak{Y}$. Suppose $G/K \in \mathfrak{Y}$. Then since $\mathfrak{Y}$ is a saturated formation, $G/\mathrm{core}_G M \in \mathfrak{Y}$. But $G/\mathrm{core}_G M$ is ϕ-free and monolithic, so $G/\mathrm{core}_G M \in \mathfrak{X}$, a contradiction to M being $\mathfrak{X}$-crucial. Thus $G/K \notin \mathfrak{Y}$. Hence H/K and, therefore, M is $\mathfrak{Y}$-crucial. Thus the $\mathfrak{X}$- and $\mathfrak{Y}$-projectors of G coincide with the $\mathfrak{X}$- and $\mathfrak{Y}$-projectors of M, respectively, but the latter coincide by induction so we are done.

The question of when $\phi R_0 \mathfrak{X}$ is a saturated formation has been answered by Cossey and McDonald [1].

Theorem 3.2 (see [1, Lemma 2.3]). *Let $\mathfrak{X}$ be a class of groups. $\mathfrak{Y} = \phi R_0 \mathfrak{X}$ is a saturated formation if and only if $\mathfrak{X}$ contains every ϕ-free monolithic group in $\mathfrak{Y}$.*

Example 3.3. If $\mathfrak{X}$ is the class of all cyclic groups of square free order, then $\phi R_0 \mathfrak{X} = \mathfrak{N}$ the class of all nilpotent groups. If $\mathfrak{X} = \{1, C_p\}$ then $\phi R_0 \mathfrak{X}$ is the class of all p-groups.

Example 3.4. Let $\mathfrak{X} = \{1, C_2, S_3\}$; then $\mathfrak{Y} = \phi R_0 \mathfrak{X}$ is not a saturated formation. If $\mathfrak{Y}$ were a saturated formation, then $C_3 \in \mathfrak{Y}$, but C_3 is ϕ-free and monolithic, so it would follow, by Theorem 3.2, that $C_3 \in \mathfrak{X}$, which is not true.

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