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LORENTZIAN MANIFOLDS OF NONPOSITIVE CURVATURE. II

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ABSTRACT. Suppose that M is a time oriented, future 1-connected, timelike and null geodesically complete Lorentzian manifold. Previously, we have shown the exponential map at any point of such a manifold embeds the future cone into M when M has nonpositive spacetime curvatures. Here we want to demonstrate that under the same hypotheses, M is homeomorphic to the product of the real line with a Cauchy hypersurface.

Let us recall briefly the main points in [F]. The basic object of study is a Lorentzian n-manifold M (signature $2-n$), which we suppose time orientable. We tacitly assume then that M is time oriented. The manifold M is called future 1-connected iff any two future-timelike (smooth) curves from p to q are homotopic through future-timelike curves with endpoints p and q. The spacetime curvatures of M are the sectional curvatures of planes spanned by a timelike and a spacelike vector. We now state Proposition 2.1 of [F]: Let M be a time oriented Lorentzian manifold with nonpositive spacetime curvatures; then the exponential map at any point of M has maximal rank on the (closed) future cone. This proposition is very useful in proving the main theorem of [F], which is stated here in the first paragraph.

Briefly our intention is to introduce the notion of a globally hyperbolic manifold, much the same as Leray did in [L], and prove that our manifolds are globally hyperbolic. Our theorem then follows from a result of Geroch, that globally hyperbolic spaces are homeomorphic to a product of the real line with a Cauchy hypersurface [G].

The first problem is to extend the ideas of timelike or null – defined only for smooth curves—to continuous curves. We use essentially the definition of Hawking and Ellis [HE] for nonspacelike curves. A continuous curve c mapping an interval of real numbers I into M is called a nonspacelike curve iff for any t in I there is an $\eta > 0$ and a normal neighborhood U of

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$c(t)$ such that for $t + \eta > s > t$ there is a future-timelike or -null curve joining $c(s)$ to $c(t)$ in U, and if $t - \eta < s < t$, there is a past-timelike or -null curve joining $c(s)$ to $c(t)$ in U. Intuitively nonspacelike curves are continuous timelike or null curves. A more thorough discussion can be found in [G] or [HE].

Let $C(p, q)$ denote the set of all equivalence classes of nonspacelike curves from p to q under the relation of reparameterization by continuous monotonic maps. Provide $C(p, q)$ with the compact open topology and observe that

$$C(p, q) = \overline{C^+(p, q)} \cup \Omega_0(p, q),$$

where $C^+(p, q)$ is the space of timelike curves from p to q and $\Omega_0(p, q)$ is the space of unbroken null geodesics from p to q without conjugate points. This follows from [HE, §4.5]. Following Leray, a time oriented manifold M is said to be globally hyperbolic iff $C(p, q)$ is empty or compact for all p and q in M. Geroch gives a geometric way of looking at the convergence of curves in $C(p, q)$ when there are no closed nonspacelike curves, compare [G].

Theorem. *Let M be a future 1-connected manifold which is timelike and null geodesically complete. Further suppose that the spacetime curvatures of M are nonpositive. Then M is globally hyperbolic.*

Proof. Suppose that $C(p, q)$ is nonempty. If $C^+(p, q)$ is empty, then $C(p, q) = \Omega_0(p, q)$. Let N_0 be the set of all null vectors u such that $c(t) = \exp(tu)$ is in $\Omega_0(p, q)$. Clearly N_0 is a discrete set because exp has maximal rank at each u in N_0 by Proposition 2.1 of [F]. Now if N_0 were an unbounded set, we could find a sequence (u_n) from N_0 such that $u_n \longrightarrow \infty$. So, given neighborhoods U_n of radius $1/n$ around q, we can find, by continuity, neighborhoods V_n of u_n such that $\exp(V_n) \subset U_n$. If we take a sequence of timelike vectors v_n in V_n, then it follows that $\exp(v_n) = q_n \rightarrow q$. In addition, v_n can be chosen arbitrarily close to u_n and so $v_n \longrightarrow \infty$ as well. Again from Proposition 2.1 of [F], it follows that exp is of maximal rank at u_1, $\exp(u_1) = q$, and so there are compact neighborhoods U of q and V of u_1 such that exp maps V diffeomorphically onto U. Further, there is a positive integer n_0 for which $n \geq n_0$ implies q_n is in U. From the main theorem of [F], exp has an inverse on the set of timelike vectors, and the restriction of this inverse to the image of timelike vectors in V must agree with the inverse of the map $\exp_V$ (restriction to V) on the image of timelike vectors in V. Hence

$$\exp^{-1}(q_n) = \exp^{-1}(\exp(v_n)) = v_n$$

for $n \geq n_0$; so the v_n are in V, contradicting the fact that $v_n \longrightarrow \infty$. As a result u_n does not go to infinity, and thus the set N_0 is bounded and so finite. In this case $C(p, q)$ is compact. If $C^+(p, q)$ were nonempty, $q = \exp(u)$ for some timelike u (again by the main theorem of [F]). First we want to prove that $\exp\colon C(0, u) \longrightarrow C^+(p, q)$ is onto, where $C(0, u)$ is the set of nonspacelike curves in the tangentspace from 0 to u. Thus for $c = \lim c_n$, c_n in $C^+(p, q)$, the curves $a_n = \exp^{-1} c_n$ are timelike curves in the tangentspace, by a similar argument as in the proof of the main theorem of [F]. It follows from the compactness of $C(0, u)$ that $a_n \longrightarrow a$, possibly passing to a subsequence, and since exp is defined on the closed cone, $\exp(a)$ makes sense. Moreover the map $\exp\colon C(0, u) \longrightarrow C(p, q)$ is continuous in the compact open topology so $\exp(a) = c$. Again $\Omega_0(p, q)$ is the image of a finite set, so $C(p, q)$ is the continuous image of a compact set and, hence, compact.

Finally, we state Geroch's result on globally hyperbolic manifolds. First, a subset S of a Lorentzian manifold is called achronal iff no p and q in S can be joined by a timelike curve. Now define $D^+(S)$ (respectively $D^-(S)$) to be the set of points p such that every past (respectively future) directed timelike curve from p without a past (respectively future) endpoint intersects S. An achronal subset S of M is called a Cauchy hypersurface iff $D^+(S) \cup D^-(S) = M$.

Theorem [G]. *M is globally hyperbolic iff M contains a Cauchy hypersurface S, in which case M is homeomorphic to the product of the real line with S.*

Theorem. *Let M be a future 1-connected manifold which is timelike and null geodesically complete. Further suppose that the spacetime curvatures of M are nonpositive. Then M is homeomorphic to the product of the real line with a Cauchy hypersurface.*

In conclusion, let us give an example of a Lorentzian manifold that is simply connected but not future 1-connected. This example is due to R. P. Geroch. Consider ordinary Minkowski three-space with coordinates x, y, and t. Let U be the open set $|t| < 1$ with the positive x-axis removed as well as the interval from -2 to 2 on the y-axis. The union of the removed sets is an infinite T-shaped figure. The set U is clearly simply connected. Choose points p and q above and below the x-axis, respectively, and join them by timelike curves straddling the x-axis. These two curves cannot be homotoped by timelike curves, since you would have to go around the part of the y-axis that has been excluded, which is impossible without introducing spacelike curves. Avez has considered a similar notion of timelike homotopy in [A] as has J. W. Smith in [S].

Finally it would be interesting to prove this theorem with future 1-connected replaced by simply connected.

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