PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 2, April 1975

wtt-COMPLETE SETS ARE NOT NECESSARILY tt-COMPLETE

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ABSTRACT. A recursively enumerable set is constructed which is complete with respect to weak truth-table reducibility but not with respect to truth-table reducibility. In contrast it is also shown that, when bounded weak truth-table reducibility is defined in the natural way, completeness with respect to this reducibility is the same as that with respect to bounded truth-table reducibility.

Truth-table (tt-) reducibility and bounded truth-table (btt-) reducibility were defined in Post [6], and weak truth-table (wtt-) reducibility was defined in Friedberg and Rogers [1]. Intuitively a set of natural numbers A is wtt-reducible to a set of natural numbers B if firstly there is an algorithm $\mathfrak{A}$ for deciding "$n \in A$" given the answers to certain questions of the form "$m \in B$?", and secondly there is a recursive function f such that when $\mathfrak{A}$ is applied to n, then every m, for which "$m \in B$?" is posed, is in $D_{f(i)}$, the finite set with canonical index $f(i)$. If there is a bound on the cardinality of $D_{f(i)}$ then we say A is *bounded* wtt-*reducible* (*or* bwtt-*reducible*) *to* B. In [1] it was shown that wtt-completeness is not the same as completeness for r.e. sets. In [2] it was shown that wtt-reducibility and tt-reducibility are not the same when restricted to r.e. sets. In [4] can be found an investigation of the wtt-degrees of r.e. sets.

As is clear from the title, the particular concern of this paper is to show that a wtt-complete set need not be tt-complete. In addition, following the method of [3], we show that a bwtt-complete set is btt-complete. Finally we remark that the simple set constructed in Post [6], and shown by Martin [5] to be complete, may or may not be tt-complete.

Unexplained terminology or notation is taken mainly from [7]. Sometimes we write $\exp(2, x)$ for 2^x.

Construction of a wtt-complete set which is not tt-complete. Let $\langle \psi_i : i \in N \rangle$ be an acceptable numbering of the binary p.r. functions. Let K denote a fixed creative set. A tt-*condition* is a Boolean combination of

Received by the editors November 30, 1973.
AMS (MOS) subject classifications (1970). Primary 02F25.

atomic statements "$n \in X$" where $n \in N$. If Φ is a tt-condition, we say Φ *is true of* $B \subset N$ if we get a true statement by substituting B for X. Adopt some fixed canonical indexing of all tt-conditions. For $B \subset N$, let $\Theta(n, B)$ be 1 or 0 according as the tt-condition with index n is true of B or not.

In stages 0, 1, 2, $\cdots$ we shall effectively enumerate $W \subset N$ and $T \subset N \times N$. The set of numbers enumerated in W by the end of stage s is denoted W_s. At the same time we shall be effectively enumerating K without repetitions, at most one member in each stage, such that there are infinitely many stages including stage 0 in which no number is enumerated in K. Also we shall be computing all the values of all the p.r. functions ψ_i.

Let k be a recursive function to be specified later. Before beginning the construction we choose a strictly increasing recursive function f such that for all n,

$$f(n) - f(n-1) \geq \exp(2, k(n) \cdot \exp(2, f(n-1))),$$

where $f(-1) = 0$ by convention. Let $I(n) = \{x: f(n-1) \leq x < f(n)\}$, $J(n) = I(0) \cup I(1) \cup \cdots \cup I(n)$ and $J(-1) = \varnothing$. W will be enumerated in such a way that if x and y are both in $I(n)$, $x < y$, and $y \in W_s$, then $x \in W_s$ also. At each stage s we define an equivalence relation E_s on N with the following properties: (i) E_s is nondecreasing with s as a set of pairs; (ii) each equivalence class of E_s is a subinterval of one of the intervals $I(n)$; (iii) if $E_s(x, y)$ and $x \in W_s$, then $y \in W_s$; (iv) there is a number j which can be found effectively from s such that $\{n\}$ is an equivalence class of E_s for all $n \geq j$.

From (ii) and (iii) we can write "$E_s \restriction (I(n) - W_s)$" without ambiguity. Let $g(n, j)$ denote the $(j+1)$th distinct value, as s increases, of

$$p(n, s) = 1 + [\text{number of equivalence classes of } E_s \restriction (I(n) - W_s)].$$

We shall ensure that if $g(n, j)$ exists, that is, if $p(n, s)$ has at least $j+1$ different values, then

$$g(n, j) \geq \exp(2, (k(n) - j) \cdot \exp(2, f(n-1))). \tag{1}$$

Later this inequality will be used to show that the number of equivalence classes of $E_s \restriction (I(n) - W_s)$ is always ≥ 1, i.e., that $I(n) - W_s$ is never empty.

The requirements that we wish to satisfy in the construction are:

$\mathfrak{C}_e$: if e is enumerated in K, then subsequently some member of $I(e)$ is enumerated in W;

$\mathfrak{T}_e$: there exists x such that either $\psi_e(e, x)$ is undefined or $\langle e, x \rangle \in T \leftrightarrow \Theta(\psi_e(e, x); W) = 0$, where e runs through N. The satisfaction of the requirements $\mathfrak{C}_e$ will ensure that $K \leq_{\mathrm{wtt}} W$, while the satisfaction of the requirements $\mathfrak{T}_e$ will ensure that $T \not\leq_{\mathrm{tt}} W$. Immediately after stage s we shall have a target value for x in $\mathfrak{T}_e$, which we denote $h(e, s)$. We now state the construction.

Stage 0. Let $h(e, 0) = 0$ for all e and $E_0(x, y)$ hold iff $x = y$. Enumerate in W each member of the range of f.

Stage $s + 1$. There are three cases.

Case 1. Some number e is enumerated in K. Let n be the least member of $I(e) - W_s$ if any. Enumerate in W each x such that $E_s(x, n)$. Let $E_{s+1} = E_s$. Let $h(y, s + 1) = h(y, s)$ for $y \leq e$, and let $h(y, s + 1) = h(y, s) + 1$ if $y > e$. Note that (1) is certainly preserved in this case.

Case 2. No number is enumerated in K, but for some number e we want to "attack" $\mathfrak{T}_e$ in the sense that $\psi_e(e, h(e, s))$ has already been computed, and either $\mathfrak{T}_e$ has not been attacked before, or since it was last attacked either $\mathfrak{T}_i$ has been attacked for some $i < e$ or some number $<e$ entered K. Choose the least such e; we attack $\mathfrak{T}_e$ as follows. Let k be the least number such that all the numbers referred to by the tt-condition with canonical index $\psi_e(e, h(e, s))$ lie in $J(k)$. Below we abbreviate $\Theta(\psi_e(e, h(e, s)), X)$ to $\Theta(X)$. We now form E_{s+1} and W_{s+1} simultaneously as follows. Firstly, let

$$E_{s+1} \restriction (J(e - 1) \cup (N - J(k))) = E_s \restriction (J(e - 1) \cup (N - J(k))).$$

Next we define $E_{s+1} \restriction I(i)$ for $e \leq i \leq k$ and simultaneously enumerate some members of $I(i)$ in W by descending induction on i. Let $i \geq e$, and $E_{s+1} \restriction I(j)$ have been defined already for all j, $i < j \leq k$. As part of the induction hypothesis we assume that we have already ensured that $\Theta(W)$ depends only on $W \cap J(i)$. Let n_0 be the greatest member of $I(i) \cap W(s)$. Let $n_1, \cdots, n_q$ be an enumeration in increasing order of those numbers $n \in I(i) - W_s$ such that for some equivalence class C of E_s, $n = \max C$. There are $\leq \exp(2, f(i - 1))$ possibilities for $W \cap J(i - 1)$. It follows that there is a subsequence $p_0, \cdots, p_r$ of $\langle n_i \colon i < q \rangle$ such that for any j, l with $j < l \leq r$, and any possible W,

$$\Theta((W - I(i)) \cup \{x \colon f(i - 1) \leq x \leq p_j\}) = \Theta((W - I(i)) \cup \{x \colon f(i - 1) \leq x < p_l\}),$$

and

$$r + 1 \geq (q + 1)/\exp(2, \exp(2, f(i - 1))). \tag{2}$$

Now define $E_{s+1} \restriction I(i)$ by letting $E_{s+1}(x, y)$ hold iff

$$\mu j[j \le r \ \&\ x \le p_j] = \mu j[j \le r \ \&\ y \le p_j].$$

At the same time enumerate in W all members of $I(i)$ which are $\le p_0$. It should be clear that $\Theta(W)$ now depends only on $W \cap J(i-1)$ because the remaining possibilities for $W \cap I(i)$ are those of the form $\{x: f(i-1) \le x \le p_j\}$.

From the way E_{s+1} is defined, it follows that $\Theta(W)$ depends only on $W \cap J(e-1)$. Thus we enumerate $\langle e, h(e, s)\rangle$ in T or not so as to ensure that $\langle e, h(e, s)\rangle \in T \leftrightarrow \Theta(W_{s+1}) = 0$. Hence $\mathcal{T}_e$ will be satisfied provided $W \cap J(e-1) = W_s \cap J(e-1)$.

Finally to complete stage $s+1$ in this case, we let $h(y, s+1) = h(y, s)$ for $y \le e$ and $h(y, s+1) = h(y, s) + 1$ for $y > e$. It should be noted that the inequality (1) is preserved in this case because of the inequality (2).

Case 3. Otherwise. Let $E_{s+1} = E_s$, $h(y, s+1) = h(y, s)$, and enumerate nothing in W or T at this stage.

This completes the construction which is obviously effective. Note from Case 2 that if $\mathcal{T}_e$ is attacked in stage $s+1$ and also in stage $s'+1$, where $s' > s$, then there exist t and $i < e$ such that $s < t < s'$, and at stage $t+1$ either i is enumerated in K or $\mathcal{T}_i$ is attacked. Let $a(i) = 1 +$ (the number of times $\mathcal{T}_i$ is attacked). By induction it is easy to show that $a(i) \le 2^{i+1}$. Now let $k(i) = 2^{i+2}$ so that $k(i) > \Sigma\{a(j): j \le i\}$. Since $p(n, s+1) \ne p(n, s)$ implies that either n is enumerated in K at stage $s+1$ or $\mathcal{T}_i$ is attacked for some $i \le n$, there are $< k(n) + 1$ distinct values of $p(n, s)$. From (1) we now have $p(n, s) \ge 2$ for all s. From Case 1 of the construction it immediately follows that each of the conditions $\mathfrak{W}_n$ is satisfied.

It remains to show that all the requirements $\mathcal{T}_e$ are met. Consider a particular e. One may easily see that $h(e, s+1) \ne h(e, s)$ for at most a finite number of values of s. Let the final value of $h(e, s)$ be denoted by $h(e)$. Suppose there is no stage $s+1$ with $h(e, s) = h(e)$ at which $\mathcal{T}_e$ is attacked. Then $\psi(e, h(e))$ is undefined. Now suppose there is such a stage $s+1$; then from the occurrence of Case 2 at that stage,

$$\langle e, h(e)\rangle \in T \leftrightarrow \Theta(\psi(e, h(e)), W_{s+1}) = 0,$$

and further $\Theta(\psi(e, h(e)), W)$ depends only on $W \cap J(e-1)$. Also, if $W_{t+1} \cap J(e-1) \ne W_t \cap J(e-1)$, then for some $i < e$ either i is enumerated in K or $\mathcal{T}_i$ is attacked at stage $t+1$, and in both these cases $h(e, t+1) = h(e, t) + 1$ which means that $t < s$. We conclude that $\mathcal{T}_e$ is satisfied.

wbtt-complete sets are btt-complete. As above let K be a fixed creative set. A sequence $\langle F_i: i \in N\rangle$ of finite sets is said to be *strongly r.e.* if

$x \in F_y$ is a binary r.e. relation and the cardinality of F_i is a recursive function of i. Call $B \subset N$ wbtt-*complete* if B is r.e. and there is a strongly r.e. sequence $\langle F_i : i \in N \rangle$ of finite sets all of the same power and a Turing reduction $\mathfrak{R}$ of K to B such that in determining via $\mathfrak{R}$ whether $i \in K$ one need pose the question "$n \in B$?" for at most those numbers n which are in F_i. In what follows, fix such B and a corresponding sequence $\langle F_i : i \in N \rangle$. It should be clear that we can simultaneously effectively enumerate K and B without repetitions, in stages $0, 1, 2, \cdots$, in such a way that whenever i is enumerated in K, then at the same stage some member of F_i is enumerated in B. We shall show that in fact B is btt-complete.

In the stages $0, 1, 2, \cdots$ we shall simultaneously enumerate a set C and form certain lists of numbers, the lists being numbered $0, 1, \cdots, m-1$ where m is the common power of all the F_i. In the kth list the $(j+1)$th number will be denoted $l(k, j)$. Of course $l(k, j)$ may not exist for all j, k, and even when it does we shall not know what it is until that stage of the construction in which it is created. By the recursion theorem we may suppose there is given for the construction of C a recursive function g such that $n \in C \leftrightarrow g(n) \in K$ for all n. We may suppose that at each stage at most one number is enumerated in K. Let K_s denote the set of numbers enumerated in K prior to stage s; similarly for B_s and C_s. The construction is as follows.

Stage s. If j is enumerated in K, then for each $k < m$ such that $l(k, j)$ already exists, enumerate $l(k, j)$ in C provided only that $l(k, j)$ is not in list k' for any k', $k < k' < m$. Next choose $k < m$ and $n < s$ such that n is not yet in C, $g(n) \notin K_{s+1}$, n is not yet in list k, and $F_{g(n)} \cap B_{s+1}$ has power k. (If no such k and n exist simply ignore this part of the instructions.) Maximize k and then minimize n to get a unique pair. Make n the next member of list k.

This completes the construction. Consider the greatest $k < m$ such that list k is infinite. Let K' consist of those j in K such that j is enumerated in K after $l(k, j)$ has been created. Then K' is creative since $K - K'$ is recursive. By choice of k there exists j_0 such that for all $j \geq j_0$, $l(k, j)$ never appears in any of the lists k', $k' > k$. Define $G_j = F_{g(l(k,j))} - B_{s+1}$, where s is the stage in which $l(k, j)$ is created. Since $g(l(k, j)) \notin K_{s+1}$, $G_j \neq \emptyset$. We claim that for all $j \geq j_0$, $j \in K$ if and only if $G_j \cap B \neq \emptyset$. Consider a particular such j. As above let s be the particular stage in which $l(k, j)$ is created. If $j \notin K'$ and $G_j \cap B \neq \emptyset$, then $l(k, j)$ will eventually appear in some list k', $k' > k$, contradicting $j \geq j_0$. Hence

$j \notin K'$ implies $G_j \cap B = \emptyset$. If $j \in K'$ then $g(l(k, j))$ will be enumerated in K at some stage $> s$, whence some member of $F_{g(l(k, j))}$ will be enumerated in B at a stage $> s$, whence $G_j \cap B \neq \emptyset$. This establishes the claim and completes the proof that B is btt-complete.

Post's simple set. Richard Ladner pointed out to me that Post's simple set S (see [6]) is wtt-complete, as can be seen from Martin's proof [5] that it is complete. He also asked whether Post's simple set is necessarily tt-complete, necessarily not tt-complete, or neither. The answer is "neither". One can construct a simultaneous effective enumeration of all the r.e. sets such that the numbering of the r.e. sets is acceptable (i.e. the enumeration is "standard"), and such that the resulting S is either tt-complete or not. To make S not tt-complete one simply refines the idea presented above for the construction of a wtt-complete but not tt-complete set. To make S tt-complete is easier, and we sketch this below.

Let $\langle W_i: i \in N \rangle$ be some given standard enumeration of the r.e. sets. Define a new standard enumeration $\langle W_i': i \in N \rangle$ as follows. For all $i \geq 1$ and $1 \leq j \leq i + 1$, let $W'_{\exp(2,2i)} = W_i$, and $W'_{\exp(2,2i)+j}$ be either the singleton $\{\exp(2, 2i + 1) + 2j + 1\}$ or $\emptyset$ according as $i \in K$ or not. Otherwise let $W_i' = \emptyset$. Recall the principle of Post's construction: W_i contributes to S the first number $> 2i$, if any, which turns up in W_i. Let S' be the simple set arising from $\langle W_i': i \in N \rangle$. If $i \geq 1$ and $i \in K$ then $F_i \subset S'$ where $F_i = \{\exp(2, 2i + 1) + 2j + 1: 1 \leq j \leq i + 2\}$. But if $i \geq 1$ and $i \notin K$, then a member of F_i can only be contributed to S' by W_k' where k has the form $\exp(2, 2l)$ and $1 \leq l \leq i$. Thus in this case $F_i \not\subset S'$. It is now obvious that $K \leq_{tt} S'$.

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