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REDUCTIVE ALGEBRAS OF COMPACT OPERATORS[1]

ROBERT A. BEKES

ABSTRACT. A closed subalgebra $\mathfrak{A}$ of the bounded operators on a Hilbert space is called reductive if every closed invariant subspace for $\mathfrak{A}$ is reducing for $\mathfrak{A}$. We show that every reductive subalgebra of the compact operators is selfadjoint.

Introduction. Let H be a Hilbert space. Let $B(H)$ denote the algebra of bounded operators on H, $K(H)$ the algebra of compact operators on H and $F(H)$ the algebra of bounded operators on H with finite dimensional range. A closed subalgebra $\mathfrak{A}$ of $B(H)$ is called reductive if every closed invariant subspace for $\mathfrak{A}$ is reducing for $\mathfrak{A}$ [i.e. if M is a closed invariant subspace for $\mathfrak{A}$ then M is also invariant for $\mathfrak{A}^* = \{T^*\colon\ T \in \mathfrak{A}\}$].

It is not known whether every weakly closed reductive algebra is selfadjoint. Using Lomonosov's theorem [5], H. Radjavi and P. Rosenthal have shown that every reductive subalgebra of $K(H)$ is selfadjoint [7]. This generalizes the following earlier result due to the author [4]: Every semisimple reductive subalgebra of the compact operators is selfadjoint.

In this paper we will show that every reductive subalgebra of $K(H)$ is semisimple and then prove Radjavi and Rosenthal's result using our techniques.

We introduce some notation. Let $\mathfrak{A} \subset B(H)$ and $M \subset H$. Then $\mathrm{cl}(\mathfrak{A})$ denotes the norm closure of $\mathfrak{A}$ in $B(H)$ and $\mathrm{cl}(M)$ denotes the closure of M in H. $\mathfrak{A}M$ denotes the span of $\{Tx\colon\ T \in \mathfrak{A}$ and $x \in M\}$. If M is a closed invariant subspace for $T \in B(H)$ then $T|M$ denotes the operator T restricted to M. If M is a closed invariant subspace for $\mathfrak{A} \subset B(H)$ then $\mathfrak{A}|M = \{T \in B(M)\colon\ T = S|M$ for some $S \in \mathfrak{A}\}$.

The main result. Let X be a Banach space. A subalgebra of $B(X)$, the bounded operators on X, is called transitive or irreducible if it has no proper

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closed invariant subspaces. The following version of Lomonosov's theorem will be used. See [5, Lemma 1].

Lemma 1. *Let* $\mathfrak{A}$ *be a transitive subalgebra of* $B(X)$ *and* T *a nonzero compact operator on* X. *Then there exist* $R \in \mathfrak{A}$ *and* $x \in X$, $x \neq 0$, *such that* $RTx = x$.

We will also need the next result, which is contained in the proof of [6, Theorem 3].

Lemma 2. *Let* $\mathfrak{A}$ *be a reductive subalgebra of* $B(H)$ *and* T *a nonzero compact operator in* $\mathfrak{A}$. *Then there exists a closed reducing subspace* M *for* $\mathfrak{A}$ *such that* $T \neq 0$ *on* M *and* $\mathfrak{A}$ *acts transitively on* M.

From these two lemmas we get

Lemma 3. *Let* $\mathfrak{A}$ *be a reductive subalgebra of* $K(H)$. *Then* $\mathfrak{A}$ *is semisimple.*

Proof. Suppose T is in the radical of $\mathfrak{A}$. If $T \neq 0$ then by Lemma 2 there exists a closed reducing subspace M for $\mathfrak{A}$ such that $T \neq 0$ on M and $\mathfrak{A}$ acts transitively on M. By Lemma 1 there exist $R \in \mathfrak{A}$ and $x \in M$, $x \neq 0$, such that $RTx = x$. This implies that 1 is in the spectrum of RT when computed in $\mathfrak{A}$. But RT is in the radical of $\mathfrak{A}$. This is a contradiction. Hence $T = 0$ and therefore $\mathfrak{A}$ is semisimple.

In the next lemma we determine the structure of the minimal closed two-sided ideals of a reductive subalgebra of $K(H)$. These ideals turn out to be the basic building blocks for reductive algebras of compact operators. An element E of a complex normed algebra $\mathfrak{A}$ is called a minimal idempotent of $\mathfrak{A}$ if $E^2 = E$ and $E\mathfrak{A}E = \{\lambda E: \lambda \text{ is complex}\}$. If $x, y \in H$ then the operator $x \otimes y$ is defined by $x \otimes y(z) = (z, y)x$ for all $z \in H$. Then $\|x \otimes y\| = \|x\| \|y\|$.

Lemma 4. *Let* $\mathfrak{A}$ *be a reductive subalgebra of* $K(H)$ *and* $E \neq 0$ *a minimal idempotent of* $\mathfrak{A}$. *Then* $\mathrm{cl}(\mathfrak{A}E\mathfrak{A})$ *is selfadjoint and isomorphic to* $\mathfrak{A}|\mathrm{cl}(\mathfrak{A}EH)$.

Proof. By induction choose $x_1, \cdots, x_k \in H$ such that $Ex_i = x_i$, $\|x_i\| = 1$ for $i = 1, \cdots, k$ and $\mathfrak{A}x_i \perp \mathfrak{A}x_j$ for $i \neq j$. If E is zero on $(\mathfrak{A}x_1 \oplus \cdots \oplus \mathfrak{A}x_k)^\perp$, then the induction stops. If E is not zero on $(\mathfrak{A}x_1 \oplus \cdots \oplus \mathfrak{A}x_k)^\perp$, then choose $z \in (\mathfrak{A}x_1 \oplus \cdots \oplus \mathfrak{A}x_k)^\perp$ such that $Ez \neq 0$. Set $x_{k+1} = Ez\|Ez\|^{-1}$. Then $Ex_{k+1} = x_{k+1}$ and $\|x_{k+1}\| = 1$. Also, since $\mathfrak{A}$ is reductive, $\mathfrak{A}x_{k+1} \perp \mathfrak{A}x_j$ for $j = 1, \cdots, k$. This process must stop in a finite number of steps since E has finite dimensional range. Suppose it stops in n steps. By [2, Lemma 1] for each $i = 1, \cdots, n$ there exists $y_i \in \mathrm{cl}(\mathfrak{A}x_i)$ such that $(x_i, y_i) = 1$ and $E|\mathrm{cl}(\mathfrak{A}x_i) = x_i \otimes y_i$. Now since the $\mathfrak{A}x_i$ are orthogonal and $E = 0$ on $(\mathfrak{A}x_1 \oplus \cdots \oplus \mathfrak{A}x_n)^\perp$ we have

$$E = \sum_{i=1}^{n} x_i \otimes y_i.$$

Set $x = x_1 + \cdots + x_n$. Then again by [2, Lemma 1] there exists $w \in \mathrm{cl}(\mathfrak{A}x)$ such that $E|\mathrm{cl}(\mathfrak{A}x) = x \otimes w$.

We claim that $\mathfrak{A}x$ is closed in H. To see this suppose that $\{T_i x\}_{i=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{A}x$. Let $S_1, \cdots, S_n \in \mathfrak{A}$. Then

$$\|(T_iE - T_jE)(S_1x_1 + \cdots + S_nx_n)\| = \left\| \sum_{k=1}^{n} (S_kx_k, y_k)(T_ix_k - T_jx_k) \right\|$$

$$\leq \sum_{k=1}^{n} \|S_kx_k\| \|y_k\| \|T_ix_k - T_jx_k\|$$

$$\leq \sum_{k=1}^{n} \|S_1x_1 + \cdots + S_nx_n\| \|y_1 + \cdots + y_n\| \|T_ix - T_jx\|$$

$$= n\|S_1x_1 + \cdots + S_nx_n\| \|y_1 + \cdots + y_n\| \|T_ix - T_jx\|.$$

Since $E = 0$ on $(\mathfrak{A}x_1 \oplus \cdots \oplus \mathfrak{A}x_n)^\perp$, $\{T_iE\}_{i=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{A}E$, which is closed in $\mathfrak{A}$. So there exists $S \in \mathfrak{A}$ such that $\|T_iE - SE\| \to 0$. Therefore $\|T_ix - Sx\| \to 0$ and so $\mathfrak{A}x$ is closed in H.

Now $E^*(x) = \sum_{i=1}^{n} y_i \otimes x_i(x) = y_1 + \cdots + y_n$. But since $\mathfrak{A}$ is reductive, $E^*|\mathfrak{A}x = w \otimes x$. So we also have $E^*(x) = w \otimes x(x) = \|x\|^2 w$. If we set $y = y_1 + \cdots + y_n$, then $y = \|x\|^2 w = nw \in \mathfrak{A}x$. By Lemma 3, $\mathfrak{A}$ is semisimple and so semiprime. Therefore by the proof of [2, Lemma 2] $\mathfrak{A}$ acts algebraically irreducibly on $\mathfrak{A}x$. So there exists $R \in \mathfrak{A}$ such that $Ry = x$. Then $Ry_i = x_i$ for $i = 1, \cdots, n$. Now let $T \in \mathfrak{A}$. Then

$$ER^*T^*Tx = x \otimes w(R^*T^*Tx) = (R^*T^*Tx, w)x$$

$$= n^{-1}(Tx, Tx)x = \sum_{i=1}^{n} n^{-1}\|Tx\|^2 x_i.$$

But we also have

$$ER^*T^*Tx = \sum_{i=1}^{n} x_i \otimes y_i(R^*T^*Tx) = \sum_{i=1}^{n} (R^*T^*Tx_i, y_i)x_i = \sum_{i=1}^{n} \|Tx_i\|^2 x_i.$$

Therefore

(1) $$n^{-1/2}\|Tx\| = \|Tx_i\| \quad \text{for all } T \in \mathfrak{A}$$

and $i = 1, \cdots, n$. From this set of equalities it follows that each of the spaces $\mathfrak{A}x_i$ is bicontinuously isomorphic to $\mathfrak{A}x$. In particular each $\mathfrak{A}x_i$ is closed in H and $\mathfrak{A}x_i$ is isometrically isomorphic to $\mathfrak{A}x_j$ for all i and j. So for all $T \in \mathfrak{A}$ we have

(2) $$\|T|\mathfrak{A}x_i\| = \|T|\mathfrak{A}x_j\|.$$

Now since $\mathfrak{A}$ is reductive, $\mathrm{cl}(\mathfrak{A}^* y_i)$ is an invariant subspace for $\mathfrak{A}$. And since $\mathfrak{A}$ acts irreducibly on $\mathfrak{A}x_i$ and $E^* y_i = y_i \neq 0$, we have $\mathrm{cl}(\mathfrak{A}^* y_i) = \mathfrak{A}x_i$. Therefore

$$F(\mathfrak{A}x_i) \subset \mathrm{cl}\ \mathrm{span}\{Sx_i \otimes T^* y_i|\mathfrak{A}x_i \colon S, T \in \mathfrak{A}\}.$$

But $Sx_i \otimes T^* y_i|\mathfrak{A}x_i = SET|\mathfrak{A}x_i$. Therefore

$$\mathrm{cl}(\mathfrak{A}E\mathfrak{A}|\mathfrak{A}x_i) = \mathrm{cl}(\mathfrak{A}|\mathfrak{A}x_i) = K(\mathfrak{A}x_i).$$

Now the $\mathfrak{A}x_i$ are orthogonal and reduce $\mathfrak{A}$, and $\mathfrak{A}E\mathfrak{A} = \{0\}$ on $(\mathfrak{A}x_1 \oplus \cdots \oplus \mathfrak{A}x_n)^\perp$, so for $T \in \mathrm{cl}(\mathfrak{A}E\mathfrak{A})$ and each i we have

$$\|T\| = \|T|\mathfrak{A}EH\| = \max\{\|T|\mathfrak{A}x_j\| \colon j = 1, \cdots, n\} = \|T|\mathfrak{A}x_i\|.$$

Therefore $\mathrm{cl}(\mathfrak{A}E\mathfrak{A})|\mathfrak{A}x_i$ is closed in $K(\mathfrak{A}x_i)$, and so

(3) $$\mathrm{cl}(\mathfrak{A}E\mathfrak{A})|\mathfrak{A}x_i = \mathfrak{A}|\mathfrak{A}x_i = K(\mathfrak{A}x_i).$$

Now let $T \in \mathrm{cl}(\mathfrak{A}E\mathfrak{A})$. Then since $\mathfrak{A}$ is reductive, by (3) we have $T^*|\mathfrak{A}x_i = (T|\mathfrak{A}x_i)^* \in \mathrm{cl}(\mathfrak{A}E\mathfrak{A})|\mathfrak{A}x_i$. Also $T^*|(\mathfrak{A}EH)^\perp = (T|(\mathfrak{A}EH)^\perp)^* = 0$. So (1) and (2) imply that $T^* \in \mathrm{cl}(\mathfrak{A}E\mathfrak{A})$. Hence $\mathrm{cl}(\mathfrak{A}E\mathfrak{A})$ is selfadjoint and isomorphic to $\mathfrak{A}|\mathrm{cl}(\mathfrak{A}EH)$.

Theorem. *Let $\mathfrak{A}$ be a reductive subalgebra of $K(H)$. Then $\mathfrak{A}$ is selfadjoint.*

Proof. Let $\mathfrak{J}$ be a maximal set of minimal idempotents of $\mathfrak{A}$ such that if $E_1, E_2 \in \mathfrak{J}$ with $E_1 \neq E_2$ then $\mathrm{cl}(\mathfrak{A}E_1\mathfrak{A}) \neq \mathrm{cl}(\mathfrak{A}E_2\mathfrak{A})$. We complete the proof in three steps.

I. We first prove that if $E_1, E_2 \in \mathfrak{A}$ with $E_1 \neq E_2$ then $\mathfrak{A}E_1H \perp \mathfrak{A}E_2H$. Let $T \in \mathfrak{A}$. Then

$$E_1TE_2 \in (\mathfrak{A}E_1\mathfrak{A})(\mathfrak{A}E_2\mathfrak{A}) \subset \mathrm{cl}(\mathfrak{A}E_1\mathfrak{A}) \cap \mathrm{cl}(\mathfrak{A}E_2\mathfrak{A}) = \{0\}$$

since the $\mathrm{cl}(\mathfrak{A}E_i\mathfrak{A})$ are distinct minimal closed ideals [1, p. 335]. Since $\mathfrak{A}$ is reductive,

$$E_1^*|\mathfrak{A}E_2H = (E_1|\mathfrak{A}E_2H)^* = 0 \quad \text{and} \quad \mathfrak{A}^*(\mathfrak{A}E_2H) \subset \mathrm{cl}(\mathfrak{A}E_2H).$$

Now let $R, S \in \mathfrak{A}$ and $u, v \in H$. Then $(RE_1u, SE_2v) = (u, E_1^*R^*SE_2v) = 0$, which completes step I.

II. Let $M =$ closed span $\{\mathfrak{A}EH: E \in \mathfrak{J}\}$. Then

$$\mathfrak{A}|M = \mathrm{cl}\left(\sum_{E \in \mathfrak{J}} \oplus \mathfrak{A}E\mathfrak{A}\right)M \quad \text{and} \quad \mathfrak{A}|M = \{T|M: T(M) = \{0\}\}.$$

To see this let $T \in \mathfrak{A}$ and $\epsilon > 0$. Since T is compact and the $\mathfrak{A}EH$ are orthogonal and reducing for $\mathfrak{A}$, we must have $\{E \in \mathfrak{J}: \|T|\mathfrak{A}EH\| \geq \epsilon\}$ a finite set, say $\{E_1, \cdots, E_n\}$. By Lemma 4 choose $T_i \in \mathrm{cl}(\mathfrak{A}E_i\mathfrak{A})$ such that $T|\mathrm{cl}(\mathfrak{A}E_iH) = T_i|\mathrm{cl}(\mathfrak{A}E_iH)$ for $i = 1, \cdots, n$. Set $T_0 = \Sigma_{i=1}^n T_i$. Then

$$T_0 \in \mathrm{cl}\left(\sum_{E \in \mathfrak{J}} \oplus \mathfrak{A}E\mathfrak{A}\right)$$

and

$$\|(T - T_0)|M\| = \max\{\|T|\mathfrak{A}EH\|: E \in \mathfrak{J}\backslash\{E_1, \cdots, E_n\}\} \leq \epsilon.$$

Therefore $T|M \in \mathrm{cl}(\Sigma_{E \in \mathfrak{J}} \oplus \mathfrak{A}E\mathfrak{A})|M$. Since $\mathrm{cl}(\Sigma_{E \in \mathfrak{J}} \oplus \mathfrak{A}E\mathfrak{A})|M = \{0\}$, it follows that $\mathfrak{A}|M^\perp = \{T|M^\perp: T(M) = \{0\}\}$.

III. We show that $M^\perp = \{0\}$ or $\mathfrak{A}|M^\perp = \{0\}$. Suppose $\mathfrak{A}|M^\perp \neq \{0\}$. Then by step II, we have that $\mathfrak{A}|M^\perp$ is a reductive subalgebra of $K(M^\perp)$, and so by Lemma 2 it is semisimple. Therefore there exists $T \in \mathfrak{A}|M^\perp$ such that the spectrum of T in $\mathfrak{A}|M^\perp$ is not zero. We can produce a nonzero idempotent E' in $\mathfrak{A}|M^\perp$ by taking the appropriate contour integral about a nonzero (isolated) point of the spectrum of T. E' must have finite dimensional range. But then by the proof of [3, Theorem 2] $\mathfrak{A}|M^\perp$ contains a minimal idempotent. Also by claim 2 we may assume that this minimal idempotent is of the form $E_0|M^\perp$, where $E_0 \in \mathfrak{A}$ and $E_0(M) = \{0\}$. Now since $\mathfrak{A}$ is reductive, $E_0TE_0 = 0$ for all $T \in \mathfrak{A}$ such that $T(M^\perp) = \{0\}$. Therefore $E_0\mathfrak{A}E_0$ is isomorphic to $E_0(\mathfrak{A}|M^\perp)E_0 = \{\lambda E_0: \lambda \text{ is complex}\}$. So E_0 is a minimal idempotent of $\mathfrak{A}$. But $\mathrm{cl}(\mathfrak{A}E\mathfrak{A})|M = \{0\}$ for all $E \in \mathfrak{J}$. So $\mathrm{cl}(\mathfrak{A}E\mathfrak{A}) \neq \mathrm{cl}(\mathfrak{A}E_0\mathfrak{A})$ for all $E \in \mathfrak{J}$. This violates the maximality of the set $\mathfrak{J}$. Hence $\mathfrak{A}|M^\perp = \{0\}$.

Therefore $\mathfrak{A} = \mathrm{cl}(\Sigma_{E \in \mathfrak{J}} \oplus \mathfrak{A}E\mathfrak{A})$, and by Lemma 4 this implies that $\mathfrak{A}$ is selfadjoint.

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DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85281

Current address: Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755