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THE KLINE 2-SPHERE CHARACTERIZATION—BY DEFINITION

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ABSTRACT. Brick partitionings are used repeatedly to prove, by definition, the classical Kline 2-sphere characterization.

Introduction. The Kline theorem states that if M is a Peano continuum not separated by the omission of any two points but separated by every 1-sphere, then M is a 2-sphere. The use of brick partitionings makes it possible to obtain this result. The bulk of the proof constitutes showing the existence of a sequence of brick partitionings $G_1, G_2, \cdots$ having the following properties: each brick of each G_i has a 1-sphere for boundary; the closure of each brick of each G_i has connected complement; each brick of each G_i may be broken into two bricks by an arc in the closure of the brick spanning the boundary of the brick and having appropriate end points such that the two new bricks may replace the original brick in G_i so that G_i retains the properties stated above; and G_{i+1} is a $1/i$-refinement of G_i.

In addition to the esthetic value of proving the Kline theorem by definition, the Zippin 2-sphere characterization and the Jordan 2-sphere characterization are obtained as corollaries. A result which is a corollary to the Kline theorem is reproved here without using the Kline theorem.

Bing proved the Kline theorem in [1]. He reproved the theorem in [3] using brick partitionings. Each proof depends on a construction which is stated in this paper as Lemma 1.

A contradiction is denoted by $\otimes$.

1. **Preliminaries.** For definitions of standard point set terms, the reader is referred to [5], while for terms concerning partitioning the reader is referred to [3].

The definition of brick partitioning is given in this section since brick partitionings are used in many of the proofs. Some necessary nomenclature precedes the definition.

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We define the *E metric* on M for $x \in M$, $y \in M$ as $E(x, y)$ is the greatest lower bound of the diameter of all connected subsets of M which contain x and y. One problem which arises when one is dealing with a set with property S occurs when one desires to partition the set in such a way that boundary has certain properties. When this happens, one would rather deal with a set which is uniformly locally connected. But if a set has property S and is connected and locally connected, it is uniformly locally connected under the E metric. The E metric also preserves the original topology. The E metric on M is denoted by $E(M; x, y)$.

Definition. A finite collection G of mutually exclusive connected open subsets of M is a *partitioning* of M if the sum of the elements of G is dense in M. If each element of G is of diameter less than ϵ, G is an *ϵ-partitioning*. If each element of G has property S, it is an *S partitioning*..

Definition. An S partitioning G of M is a *brick partitioning* if:

(a) each domain containing a point of M which is a limit point of each of two elements of G also contains a point of M which is a limit point of each of these same two elements of G but of no other element of G,

(b) each element of G is uniformly locally connected under $E(M; x, y)$,

(c) each boundary point in M of an element of G is a boundary point of another element of G.

2. **The Kline sphere theorem.** The pattern of proof used to show that a Peano continuum D is by definition homeomorphic to a closed euclidean 2-cell E is almost universal. A dense subset M of D, a dense subset K of E and a function f are determined such that $f: M \to K$ is 1-1, and onto, while both f and f^{-1} are uniformly continuous. It is well known that f can be extended to establish a homeomorphism between D and E. For example, see [5, Lemma 4.1, p. 87] and the theorems following it. The variations in the proof are in determining M. Brick partitionings are found to be quite helpful in this endeavor.

We begin by proving several lemmas. The first one Bing showed implied the Kline theorem via Zippin's characterization of a closed 2-cell.

Lemma 1. *Let M be a Peano continuum which is not separated by the omission of any two of its points and α an arc of M with end points p and q which irreducibly separates M. Then there exists a 1-sphere J in M which intersects α only in p and q while $M - J$ is connected and has property S.*

Proof. See [3, Theorem 20].

Note. If α separates space then some subarc of α irreducibly separates space.

Lemma 2. *Let M be a Peano space. Then for every $\epsilon > 0$ and $p \in M$ there is a domain D in M having property S such that $p \in D$, while $\bar{D}$ is a Peano continuum, has connected complement and is of diameter less than ϵ.*

If M has no local cut points then $\bar{D}$ has no local cut points.

Proof. See [4, Lemma 2.5].

The following lemma is a trivial corollary to the Kline theorem. See [4, Lemma 2.1]. It is reproved here without using the Kline theorem.

Lemma 3. *Let M be a Peano space with no local cut points. Suppose D is a domain whose closure is a proper subset of M, is a Peano continuum, has connected nonempty complement and has the property that every 1-sphere in $\bar{D}$ separates M. Then $\bar{D}$ contains a 1-sphere J such that $\bar{D} - J$ is connected and has property S while* $\operatorname{Bd} \bar{D} \subset J$.

Proof. Let $p \in \operatorname{Bd} \bar{D}$. By Lemma 2 there is a domain C in $\bar{D}$ having property S such that $p \in C$, while $\bar{C}$ is a Peano continuum, has no local cut points and has connected nonempty complement relative to $\bar{D}$. Let α be an arc in $D - C$ spanning $\bar{C}$. Since C has property S there is an arc β in $\bar{C} \cap D$ spanning the end points of α. Now $\alpha \cup \beta = J_1$, a 1-sphere in D. J_1 separates space; each component intersecting $\bar{C}$. If β does not separate $\bar{C}$, then since C has property S, every point of $M - J$ may be joined to p by an arc in $M - J$. Thus β separates $\bar{C}$. By Lemma 1 there is a 1-sphere J_2 of $\bar{C}$ which does not separate $\bar{C}$. Since J_2 separates M, $\operatorname{Bd} \bar{C} \subset J_2$. Thus $p \in J_2$. Since $\bar{C}$ is a neighborhood of p there is an arc of J_2 separating $\bar{D}$. Finally, by Lemma 1, there is a 1-sphere J of $\bar{D}$ such that $\bar{D}$ does not separate $\bar{D}$. Since J separates M, $\operatorname{Bd} \bar{D} \subset J$.

The following lemma allows us to utilize Lemmas 2 and 3.

Lemma 4. *Let M be a Peano continuum which is separated by every 1-sphere but not separated by the omission of any pair of points. Then M has no local cut points.*

Proof. By [5, Corollary 3.32a, p. 86], M is cyclic. Assume p is a cut point of the neighborhood U of p in M. By Lemma 2 there is a neighborhood V of p such that $\bar{V} \subset U$, while $\bar{V}$ is a Peano continuum, having connected complement. Since $\bar{V}$ contains a 1-sphere, and any such 1-sphere is contained in a single cyclic element of $\bar{V}$, $\bar{V}$ contains a nondegenerate cyclic element K.

We now show that $\overline{V} = K$. Suppose $K \neq V$. Let J be any 1-sphere of K. J separates M. By [5, Theorem 3.24, p. 83] each component of $\overline{V} - K$ has exactly one limit point in K. Thus each component of $\overline{V} - K$ meets Bd V. Thus $(M - \overline{V}) \cup (\overline{V} - K) = M - K$ is connected. Hence one of the components of $M - J$ is contained in K. Now each pair of points of K belongs to a 1-sphere J' of K and a component C of $M - J'$ is contained in K, where Bd $C = J'$. Then no two points of K separate K.

There is an arc of $M - K$ spanning Bd K with end points x and y. There is an arc α of $\overline{V}$ containing x and y and also a point of Int(K). The union of these two arcs forms a 1-sphere J'. J' separates M. Since $M - K$ is connected, α separates K. Thus by Lemma 1, there is a 1-sphere J'' of K which does not separate K. Thus Bd $K \subset J''$. Since no subarc of J'' separates M, Bd $K = J''$. Thus there is an arc β of J'' such that each point of β is a limit point of a component of $\overline{V} - K$. Thus $\overline{V}$ contains an uncountable number of disjoint open sets. ⊗ Thus $\overline{V} = K$. Then $\overline{V}$ is cyclic. Hence p is not a cut point of $\overline{V}$. ⊗

Theorem. *If M is a Peano continuum which is not separated by the omission of any 2 points but is separated by every 1-sphere, then M is a 2-sphere.*

Proof. The proof will be done in sectionalized form.

(1) Each 1-sphere J of M separates M into 2 components R_1 and R_2 each of which has property S, while Bd $R_1 = J =$ Bd R_2.

Proof. Let J be any 1-sphere of M. Let R_1 and R_2 denote 2 complementary domains of J. If Bd $R_1 \neq J$ then some arc of J separates M. By Lemma 1 there is a 1-sphere which does not separate space. ⊗ Thus Bd $R_1 = J =$ Bd R_2. Suppose there is a third complementary domain R_3 of J. Let α be an arc of $\overline{R}_1$ spanning J and β an arc of $\overline{R}_2$ spanning J. Now $\alpha \cup \beta \subset J'$ a 1-sphere contained in $\alpha \cup \beta \cup J$. Let $T = J - J'$. Each point of $R_1 - J'$ may be joined to T or the arc $\overline{R}_1 - J'$ separates M. Then every point in $M - J'$ may be joined to T by an arc. Since $T \subset$ Bd R_3, $M - J'$ is connected. ⊗ Thus $M - J = R_1 \cup R_2$ while Bd $R_1 = J =$ Bd R_2. By Lemmas 4 and 1, R_1 and R_2 each have property S.

(2) $\overline{R}_1(\overline{R}_2)$ has no local cut points.

Proof. Let p be a local cut point of $\overline{R}_1$. By Lemma 4, $p \in J$. Also p separates an arc of J in some neighborhood V of p. Let U be a neighborhood of p in M such that $U \cap \overline{R}_1 \subset V$. By Lemma 2 there is a neighborhood D of p such that $\overline{D}$ is a Peano continuum, $\overline{D} \subset U$, while $\overline{R}_1 - \overline{D}$ is connected.

By Lemma 3, $\overline{D}$ contains a 1-sphere J' such that $\overline{D} - J'$ is connected while Bd $\overline{D} \subset J'$. Thus Bd $\overline{D} = J'$(rel M); hence $p \in J'$. Thus, some are of $J' \cap \overline{R}_1$ spans $J - p$. But $J' \cap \overline{R}_1 \subset V$. ⊗ Thus p is not a cut point of V. This shows that $\overline{R}_1(\overline{R}_2)$ has no local cut points.

The proof will consist of showing that $\overline{R}_1$ is a closed 2-cell. For the rest of the paper R_1 will be denoted by R.

(3) For every $\epsilon > 0$ there is a brick ϵ-partitioning G of R such that the boundary of each brick in G consists of a finite number of 1-spheres.

Proof. Let $\epsilon < .01$ diam J. By [3, Theorem 8] there is a brick ϵ-partitioning of R. Let $g \in G$ and C be any complementary domain of g (rel M). Let α be an arc of $\overline{C} \cap R$ spanning $\overline{g}$. Since g has property S there is an arc β of $\overline{g}$ spanning Bd g such that $\alpha \cup \beta = J_1$ a 1-sphere of R. J_1 separates M. Also each component of $M - J_1$ intersects g. Thus β irreducibly separates $\overline{g}$. By Lemma 1 there is a 1-sphere J_2 of $\overline{g}$ containing the end points of β which does not separate $\overline{g}$. If $J_2 \cap g \neq \emptyset$ then some arc α in $J_2 \cap \overline{g}$ spanning Bd g does not separate $\overline{g}$. Since each brick of G has property S, α can be shown to be an arc of a 1-sphere J_3 where $J_3 \cap \overline{g} = \alpha$. But then α must separate $\overline{g}$, or J_3 does not separate M. Thus $J_2 \cap g = \emptyset$, hence $J_2 \subset$ Bd g. Since J_2 has exactly 2 complementary domains and $J_2 \cap \overline{C} \neq \emptyset$, Bd $C \subset J_2$. Finally Bd $C = J_2$. Thus the boundary of each brick g of G consists of a finite number of 1-spheres.

(4) G may be refined so that Bd g(rel M) consists of exactly one 1-sphere.

Proof. Let G be any brick partitioning of R. Denote the maximum number of boundary 1-spheres of the bricks of G by $n(G)$. If $n(G) = 1$ we have the desired result. Suppose $k > 1$ and that the proposition is true for all G with $n(G) < k$. Let G be a brick partitioning of R with $n(G) = k$. Choose $g \in G$ such that g has k boundary 1-spheres. Let J_1 and J_2 denote any 2 boundary spheres of g. There is an arc α of $\overline{g}$ which spans Bd g having one end point on J_1 and the other end point on J_2. By Lemma 1, α does not separate $\overline{g}$. Let β be an arc of $\overline{g} - \alpha$ which spans Bd g having one end point on J_1 and the other end point on J_2. Let $\delta <$ .01 $d(\alpha, \beta)$. Using [3, Theorem 8] we may obtain a brick δ-partitioning H of g which may be substituted for g in G to retain a brick partitioning of R. Consolidate all the bricks h of H for which $\overline{h} \cap \beta \neq 0$. Call this brick h'. Let h_1 be the consolidation of all the bricks h of H which lie in the component of $g - h'$ which intersects α. Let h_2 be the consolidation of h' and all the bricks of H not in h_1. Thus, $\overline{h}_1 \cup \overline{h}_2 = \overline{g}$. Now g is replaced in G by h_1 and h_2. The refinement of g into h_1 and h_2 has created 2 new boundary 1-spheres: the boundary 1-sphere of $h_1(h_2)$ which separates

$h_1(h_2)$ from $h_2(h_1)$. However J_1 and J_2 are no longer boundary 1-spheres in this refinement. Thus, h_1 and h_2 have at most $(k-1)$ boundary 1-spheres. Proceeding in this manner with each brick of G which has k boundary 1-spheres, one arrives at a refinement K of G such that $n(K) < k$. By the induction hypothesis K may be refined to a partitioning G' for which $n(G') = 1$.

(5) Let $g \in G$ and α an arc of $\bar{g}$ spanning Bd g such that each end point is the limit point of exactly one other brick of G. Denote the two arcs of Bd g spanning the end points of α by β and γ. Then $\bar{g} - (\alpha \cup \beta \cup \gamma)$ has exactly two components C_1 and C_2 where Bd $C_1 = \alpha \cup \beta$ and Bd $C_2 = \alpha \cup \gamma$. Furthermore a brick refinement of G having all the properties of G is obtained by substituting C_1 and C_2 for g in G.

Proof. $M - g$ is a complementary domain of the boundary 1-sphere of g; thus has property S. Hence there is an arc in $M - g$ spanning Bd g with the same end points as α. Then α belongs to a 1-sphere whose intersection with $\bar{g}$ is α. This shows that α separates $\bar{g}$.

Let C_1 be the component of $\alpha \cup \beta$ which does not intersect γ. Then Bd $C_1 = \alpha \cup \beta$. Let C_2 be the component of $\alpha \cup \gamma$ which does not intersect β. Then Bd $C_2 = \alpha \cup \gamma$. By Lemma 3, $\bar{C}_1$ contains a 1-sphere J' such that $\bar{C}_1 - J'$ is connected and has property S, while Bd $\bar{C}_1 \subset J'$. Then Bd $\bar{C}_1 = J'$, while $\bar{C}_1 - J' = C_1$ has property S. Similarly C_2 has property S.

Suppose $\bar{g} - (\alpha \cup \beta \cup \gamma)$ has a third complementary domain C_3. Then $C_3 \cap \beta \neq \emptyset$ and $C_3 \cap \alpha \neq \emptyset$. But then α does not separate C_1 from C_2 in $\bar{g}$. Any other complementary domain of $\bar{g} - (\alpha \cup \beta \cup \gamma)$ must also intersect β and α. Thus α does not separate $\bar{g}$. $\otimes$ This shows that $C_1 \cup C_2 = \bar{g} - (\alpha \cup \beta \cup \gamma)$.

In order for $C_1(C_2)$ to be bricks, each must be uniformly locally connected under the metric $E(R;\, x,\, y)$ discussed in §1. Assume C_1 is not uniformly locally connected under the metric $E(R;\, x,\, y)$. Then there is a point $p \in \alpha$ and $\epsilon > 0$ such that there exist two sequences of points $\{x_n\}$ and $\{y_n\}$ in C_1, both converging to p, such that $E(R;\, x_n,\, y_n) > \epsilon$ for all n. By Lemma 2 there is a domain D in $\bar{C}_1$ such that $p \in D$, while $\bar{D}$ is a Peano continuum, has connected complement and is of diameter less than ϵ. By Lemma 3, $\bar{D}$ contains a 1-sphere J' such that $\bar{C}_1 - J'$ is connected while Bd $\bar{D} \subset J'$. Since $p \in \bar{D}$ and J' does not separate $\bar{D}$, $p \in J'$. Indeed, there is a subarc of α contained in J'. But then for some n both x_n and y_n belong to $\bar{D} - J'$. Thus $E(R;\, x_n,\, y_n) < \epsilon$. $\otimes$

By the definition of brick partitioning and the restriction on the end points of α, C_1 and C_2 may be substituted for g in G to obtain a brick refinement of having all the properties of G.

(6) $\overline{R}$ is a closed 2-cell.

Proof. Note that $\overline{g}_i \cap \overline{g}_j$ and $\overline{g}_i \cap J$ is empty, an arc, or a finite collection of arcs. Let K be the set consisting of the interiors of the arcs of $\overline{g}_i \cap \overline{g}_j$ and $\overline{g}_i \cap J$, calling these arcs 1-cells; the end points of these arcs, calling these 0-cells; and the elements of G, calling these 2-cells. K is called a 2-complex and is denoted by $K(G)$.

By induction and (5) for each 2-complex L associated with a brick partitioning H of a closed 2-cell E satisfying (4), there is a brick partitioning G of R satisfying (4) such that $K(G)$ is isomorphic to $L(H)$; i.e., there is a 1-1 correspondence between the cells of K and the cells of L which preserves dimensionality and boundary relationships both ways.

For each $g \in G$ no two bricks $g_1, g_2 \in G$ have the property that $\overline{g}_i \cap \overline{g}$ separates $\overline{g}_{3-i} \cap \overline{g}$ in Bd g for $i = 1, 2$. Thus one may always find a brick $g_3 \in G$ such that g may be consolidated with g_3 to form a brick satisfying (4). By induction and (5) for each $K(G)$ there is an $L(H)$ which is isomorphic to $K(G)$.

Let ϵ_i be a monotone decreasing sequence of real numbers converging to zero where $\epsilon_1 < .01$ diam J. Let G_1' be a brick ϵ_1-partitioning of R; $L(H_1')$ a 2-complex isomorphic to $K(G_1')$; H_1 a brick ϵ_1-refinement of H_1'; and G_1 a brick ϵ_1-refinement of G_1' so that $K(G_1)$ is isomorphic to $L(H_1)$. Suppose $K(G_i)$ and $L(H_i)$ have been obtained where $K(G_i)$ is isomorphic to $L(H_i)$, where G_i is a brick ϵ_i-partitioning of R and a refinement of G_{i-1}, while H_i is a brick ϵ_i-partitioning of E and a refinement of H_{i-1}. Let G_{i+1}' be a brick ϵ_{i+1}-refinement of G_i; $L(H_{i+1}')$ a 2-complex isomorphic to $K(G_{i+1})$, where H_{i+1}' is a refinement of H_i; H_{i+1} a brick ϵ_{i+1}-refinement of H_{i+1}; and $K(G_{i+1})$ a 2-complex corresponding to $L(H_{i+1})$, where G_{i+1} is a refinement of G_{i+1}.

Let X denote the set of points which are 0-cells of at least one $K(G_i)$, $i = 1, 2, \cdots$. Let Y denote the set of points which are 0-cells of at least one $L(H_i)$, $i = 1, 2, \cdots$. Note that X is dense in $\overline{R}$ while Y is dense in E. Let $f: X \to Y$ where $f(x)$ is defined by the isomorphisms obtained.

Now f and f^{-1} are both uniformly continuous. Let $\epsilon > 0$ and choose i such that $2\epsilon_i < \epsilon$. By way of contradiction one may show that there is a $\delta > 0$ such that if x and y are points of $\overline{R}(E)$ of distance apart less than δ, then x and y belong to or are on the boundary of the same or intersecting 2-cells. If two points of $X(Y)$ are of distance apart $< \delta$, the points corresponding to them under f (f^{-1}) in $X(Y)$ are in or on the boundary of the same or intersecting 2-cells in $K(G_i)$ $(L(H_i))$, so that their distance apart is $< \epsilon$. Thus, f and f^{-1} are uniformly continuous.

Following are two corollaries. The first is a direct result of the Theorem. The second corollary follows from the first corollary.

Corollary 1. *A Peano continuum C which satisfies the following three conditions is a 2-sphere:*

(a) *C contains at least one 1-sphere.*

(b) *Every 1-sphere of C separates C.*

(c) *No arc which lies on a 1-sphere of C separates C.*

Corollary 2. *A Peano continuum which contains at least one 1-sphere and which satisfies the Jordan curve theorem is a 2-sphere.*

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