PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 2, April 1975

RINGS OF EQUIVALENT DOMINANT AND CODOMINANT DIMENSIONS

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ABSTRACT. Continuing an earlier examination of the codominant dimension of rings and modules, a categorical characterization is given for rings of equivalent dominant and codominant dimensions. Specifically, the question is reduced to when ${}_RR$ and the left minimal injective cogenerator ${}_R\mathfrak{U}$ can be used as test modules respectively for the dominant dimension of the projective modules and the codominant dimension of the injective modules. These conditions are in turn characterized by when the injective projective modules are Σ-injective. Also, a new and shortened version is given for the proof that the codominant dimension of the injective modules is equal to the dominant dimension of the projective modules.

Throughout, R is an associative ring with unity and all modules considered are unital. We denote a minimal injective cogenerator for ${}_R\mathfrak{M}$ by ${}_R\mathfrak{U}$, such being isomorphic to an injective hull of a direct sum of a complete representative class of simple left R-modules. For a left perfect ring R and a left R-module ${}_RM$ we define the *codominant dimension* of ${}_RM$, denoted codom $\dim_R M$, to be greater than or equal to n if for a minimal projective resolution $\cdots \to P_n \to \cdots \to P_1 \to M \to 0$, P_i is injective for $i = 1, 2, \cdots, n$. The left codominant dimesnion of R is defined to be codom dim ${}_R\mathfrak{U}$. The *dominant dimension* of ${}_RM$ is defined in a dual manner, and the left dominant dimension of R is dom $\dim_R R$. Further, for R left perfect, we define codom $\dim_R \mathcal{I}$ to be $\inf\{\text{codom}\dim_R E\colon {}_RE \text{ injective}\}$. Similarly, dom $\dim_R \mathcal{P}$ is given by $\inf\{\text{dom}\dim_R P\colon {}_RP \text{ projective}\}$. Finally, we say that ${}_RE$ (${}_RP$) is a *test module* for ${}_R\mathcal{I}$ (${}_R\mathcal{P}$) provided

$$\text{codom}\dim_R E = \text{codom}\dim_R \mathcal{I} \quad (\text{dom}\dim_R P = \text{dom}\dim_R \mathcal{P}).$$

In [2] we showed that dom $\dim_R \mathcal{P}$ = codom $\dim_R \mathcal{I}$. As an application it can

Presented to the Society, January 25, 1973; received by the editors April 16, 1973.

AMS (MOS) subject classifications (1970). Primary 16A60; Secondary 16A50, 16A52.

Key words and phrases. Perfect rings, injective projective modules, codominant and dominant dimensions, Σ-injectivity.

be shown that the left codominant dimension of a left artinian ring is equivalent to the ring's left dominant dimension. In fact, this follows from the result that the minimal injective cogenerator ${}_R\mathfrak{U}$ and the left regular module ${}_RR$ can be used as test modules for ${}_R\mathcal{I}$ and ${}_R\mathcal{P}$, respectively. In this paper we propose to give a categorical characterization of those rings for which this can be done.

As shall be seen, this is closely related to when the direct sum of projective injective modules is injective and rings with ascending condition on annihilators (see [3]). To facilitate this discussion we provide the following definitions.

An injective module ${}_RM$ is *Σ-injective* provided $\bigoplus\Sigma_\Omega M$ is injective for all Ω. Faith [3, Proposition 3.3] has shown that it is sufficient to consider Ω of countable cardinality in testing for Σ-injectivity. As the categorical dual concept we have *π-projectivity.*

For a left R-module ${}_RM$ and a subset $X \subseteq M$ we set

$$\mathrm{Ann}_R(X) = \{r \in R\colon rX = 0\}.$$

The collection of all such left ideals for a given ${}_RM$ is denoted by $\mathfrak{A}\mathrm{nn}_R(M)$. For the special case of ${}_RM = {}_RR$, $\mathfrak{A}\mathrm{nn}_R(R)$ is called the set of left *annulets* of R.

1. **The left regular module ${}_RR$ as a test module.** In this section we determine when ${}_RR$ is a test module. Clearly this is a triviality when $\mathrm{dom\ dim}_RR = 0$.

Theorem 1.1. *Let R be a left perfect ring of positive left dominant dimension. Then ${}_RR$ is a test module for the dominant dimension of the projective modules if and only if each injective projective left module is Σ-injective.*

Proof. ($\Rightarrow$) It suffices to show that $\bigoplus\Sigma_\Omega P_\alpha$ is injective, where $P_\alpha \cong Re$ for each $\alpha = \Omega$ is an indecomposable injective projective. Further, we may assume that the cardinality of Ω is $\aleph_0$.

Let $0 \to \bigoplus\Sigma_\Omega P_\alpha \to E(\bigoplus\Sigma_\Omega P_\alpha)$ be the canonical injection. Since P_α is injective, $\bigoplus\Sigma_{\Omega'} P_\alpha$ is isomorphic to a direct summand of $E(\bigoplus\Sigma_\Omega P_\alpha)$ for each finite subset Ω' of Ω. Moreover, with $E(\bigoplus\Sigma P_\alpha)$ projective, $E(\bigoplus\Sigma_\Omega P_\alpha) \cong \bigoplus\Sigma_\Gamma Re_\beta$, where each Re_β is injective. Whereas $\bigoplus\Sigma_{\Omega'} P_\alpha$ is isomorphic to a direct summand of $\bigoplus\Sigma_\Gamma Re_\beta$ for each finite subset Ω' of Ω, $P_\alpha \cong Re_\beta$ for infinitely many $\beta \in \Gamma$. Thus, $\bigoplus\Sigma_\Omega P_\alpha$ is injective.

($\Leftarrow$) For ${}_RP$ projective we have ${}_RP \cong \bigoplus\Sigma_{\Omega} Re_{\alpha}$. We consider a minimal injective resolution $0 \to Re_{\alpha} \to E_{1\alpha} \to \cdots \to E_{n\alpha} \to \cdots$ for each $\alpha = \Omega$. Then, with dom $\dim_R R \geq n$,

$$0 \to \bigoplus\sum_{\Omega} Re_{\alpha} \to \bigoplus\sum_{\Omega} E_{1\alpha} \to \cdots \to \bigoplus\sum_{\Omega} E_{n\alpha}$$

provides the desired portion of a minimal injective resolution of $\bigoplus\Sigma Re_{\alpha}$.

Note that the sufficiency part of the proof depended only on the dominant dimension of the projectives being nonzero.

Corollary 1.2. *If R is a left perfect ring of positive left dominant dimension and ${}_RR$ is a test module for ${}_R\mathcal{P}$, then R is semiprimary.*

Proof. In this setting $E({}_RR)$ is Σ-injective. Then, by Corollary 3.4 of [3], R satisfies a.c.c. on left annulets. But then by Proposition 4.1 of [3], R is semiprimary.

Corollary 1.3. *A ring R is left self-injective, left perfect and has ${}_RR$ as a test module for ${}_R\mathcal{P}$ if and only if R is QF.*

Proof. For such a ring every projective left module is injective and so, the ring is QF (see [3]).

In [3] Faith points out that a.c.c. on left annulets of R does not imply Σ-injectivity of $E({}_RR)$, although the converse does hold. We provide a case where this is true.

Proposition 1.4. *Let R be a left perfect ring of positive left dominant dimension. Then, $\mathfrak{Ann}_R(E({}_RR)) = \mathfrak{Ann}_R(R)$.*

Proof. Let M be a subset of $E({}_RR) \cong \bigoplus\Sigma_{\Omega} Re_{\alpha}$. Set $M_{\alpha} = \rho_{\alpha}(M) \subseteq Re_{\alpha}$, where $\rho_{\alpha}\colon E({}_RR) \to Re_{\alpha}$ is the canonical projection. Then, $\{r \in R\colon rM = 0\}$ $= \{r \in R\colon r(\bigcup_{\Omega} M_{\alpha}) = 0\}$.

Corollary 1.5. *Let R be left perfect and* dom $\dim_R R \neq 0$. *Then the following are equivalent*:

(1) *${}_RR$ is a test module for ${}_R\mathcal{P}$*;

(2) *R satisfies a.c.c. on left annulets*;

(3) *For each left ideal I there corresponds a finitely generated subideal I' having the same right annihilator as I.*

Proof. In light of (1.4) this is a restatement of Proposition 3.3 of [3].

2. **The minimal injective cogenerator as a test module.** First we note that rings in which codom $\dim_R \mathcal{J} \neq 0$ have the same Σ-injectivity property as noted in the previous section.

Proposition 2.1. *Let R be a left perfect ring for which* codom $\dim_R \mathcal{I} \neq 0$. *Then, each injective projective left module is Σ-injective.*

Proof. Theorem 3.1 of [2] states that codom $\dim_R \mathcal{I}$ = dom $\dim_R \mathcal{P}$. Consequently, this is immediate from the remark following (1.1).

In establishing our next theorem we shall make use of the following observation.

Proposition 2.2. *If R is a left perfect ring and $_RP$ is an injective projective module which is Π-projective, then $_RP$ is Σ-injective. For R perfect the converse also holds.*

Proof. It suffices to show that $\oplus\Sigma_\Omega Re$ is injective for some infinite index set Ω, Re an indecomposable injective projective left R-module. For any infinite set Γ, $\Pi_\Gamma Re$ is projective and so, isomorphic to $\oplus\Sigma_\Delta Re_\alpha$ for some infinite set Δ. Suppose that only finitely many of the Re_α's are isomorphic to Re, say n in all. The module $\Pi_\Gamma Re$ has a submodule isomorphic to $\oplus\Sigma^{n+1} Re$, so $\oplus\Sigma_\Gamma Re_\alpha$ has a direct summand isomorphic to $\oplus\Sigma^{n+1}Re$. This is impossible and so infinitely many of the Re_α's are isomorphic to Re. Thus, $\oplus\Sigma_\Gamma Re_\alpha$ has a direct summand of the form $\oplus\Sigma_\Omega Re$ with Ω infinite.

For R perfect, every nonzero module has a nonzero socle. We again let Re be an indecomposable injective projective left R-module and consider $\Pi_\Omega Re$, Ω infinite. Every simple submodule of $\Pi_\Omega Re$ is isomorphic to Re's unique simple submodule. Hence, $\Pi_\Omega Re \cong \oplus\Sigma_\Gamma Re$ and $\Pi_\Omega Re$ is projective.

So as to relate the codominant dimension of the minimal injective cogenerator $_R\mathfrak{U}$ to the codominant dimension of an arbitrary injective module, we shall apply the next result.

Proposition 2.3. *Let R be left perfect with* codom $\dim_R \mathcal{I} \neq 0$ *and* codom $\dim_R \mathfrak{U} \geq n$. *Then, if $\cdots \to P_n \to \cdots \to P_1 \to {}_R\mathfrak{U} \to 0$ is a minimal projective resolution of $_R\mathfrak{U}$, P_i, $1 \leq i \leq n$, generates the corresponding ith term in the minimal projective resolution of an arbitrary injective left R-module.*

Proof. By Theorem 3.1 of [2], (1.1) and (2.1), we note that the hypothesis of (1.2) is satisfied. Thus, R is perfect. For each injective $_RE$,

$$_RE \cong E\left(\oplus\sum_{\Omega_1} \frac{Re_1}{Je_1}\right) \oplus \cdots \oplus E\left(\oplus\sum_{\Omega_m} \frac{Re_m}{Je_m}\right),$$

where $e_1, \cdots, e_m$ belong to a basic set of primitive idempotents for R. We shall establish (2.3) for $E(\bigoplus\Sigma_{\Omega_1}S)$, where $S \cong Re_1/Je_1$.

First observe that $E(\bigoplus\Sigma S)$ is isomorphic to a direct summand of $\Pi_{\Omega_1}E(S)$. Next, let $P_{1n} \to \cdots \to P_{11} \to E(S) \to 0$ be the first n terms in a minimal projective resolution of $E(S)$.

By (2.1) and (2.2),

$$\prod_{\Omega_1} P_{1n} \to \cdots \to \prod_{\Omega_1} P_{11} \to \prod_{\Omega_1} E(S) \to 0$$

are the first n terms in a projective resolution of $\Pi E(S)$. Moreover, the ith term in a minimal projective resolution of $E(\bigoplus\Sigma_{\Omega}S)$ is isomorphic to a direct summand of $\Pi_{\Omega_1}P_{1i}$, $1 \le i \le n$. Also, such a summand is of the form $\bigoplus\Sigma_{\Gamma}Re_{\beta}$ with each Re_{β} isomorphic to a direct summand of P_{1i}. Thus, P_{1i}, and so P_i, generates $\bigoplus\Sigma_{\Gamma}Re_{\beta}$.

Combining (2.1) and (2.3) along with the results of the previous section we have the following characterization of when the minimal injective cogenerator ${}_R\mathfrak{U}$ is a test module.

Theorem 2.4. *Let R be left perfect and of positive left codominant dimension. Then the following are equivalent:*

(1) *${}_R\mathfrak{U}$ is a test module for ${}_R\mathfrak{I}$.*

(2) *Each injective projective left module is Σ-injective.*

(3) *Each injective projective left module is Π-projective.*

(4) *R has a.c.c. on left annulets and $E({}_RR)$ is projective.*

(5) *${}_RR$ is a test module for ${}_R\mathfrak{P}$ and* dom $\dim_R R \neq 0$.

Proof. (1)$\Leftrightarrow$(2). This is (2.1) and (2.3).

(1), (2)$\Leftrightarrow$(3). By Theorem 3.1 of [2], (1.1) and (1.2), we have sufficiency. Necessity is (2.2).

$\Leftrightarrow$(4). This equivalency is given by Theorem 3.1 of [2], (1.4) and Proposition 3.3 of [3].

$\Leftrightarrow$(5). See (1.1).

3. **When codominant and dominant dimension are equivalent.** First we restate the main result of [2], replacing the long tedious proof given there by a greatly clarified version.

Theorem 3.1. *Let R be left perfect. Then* codom $\dim_R\mathfrak{I} \ge n$ *if and only if* dom $\dim_R\mathfrak{P} \ge n$.

Proof. ($\Rightarrow$) We induct on n. Let ${}_RP$ be projective, and suppose that codom $\dim_R \mathcal{I} \geq k+1$ and that dom $\dim_R \mathcal{P} \geq k$. Consider

$$\begin{array}{ccccccccc} & & & & & & & & P_{k+1} \\ & & & & & & & & \downarrow \\ & & & & & & & & \vdots \\ & & & & & & & & \downarrow \\ & & & & & & & & P_1 \\ & & & & & & & & \downarrow \phi_1 \\ 0 & \longrightarrow & P & \longrightarrow & E_1 & \xrightarrow{\psi_1} & \cdots & \longrightarrow & E_{k+1} \\ & & & & & & & & \downarrow \\ & & & & & & & & 0 \end{array}$$

where the row is a minimal injective resolution of ${}_RP$ and the column a minimal projective resolution of E_{k+1}. Since $\mathrm{Hom}_R(\_, M)$ and $\mathrm{Hom}_R(N, \_)$ are exact functors whenever ${}_RM$ is injective and ${}_RN$ is projective, the above diagram results in the commutative diagram given by Figure 1, in which all the columns are exact and the first $k+1$ rows are exact and the last row semi-exact. To show that E_{k+1} is projective it will suffice to show that the identity map $\iota \in \mathrm{Hom}_R(E_{k+1}, E_{k+1})$ has a preimage under $\mathrm{Hom}_R(E_{k+1}, P_1) \to \mathrm{Hom}_R(E_{k+1}, E_{k+1})$, since in such a case E_{k+1} will be isomorphic to a submodule of P_1. Using Figure 1 we diagram chase $\iota = \mathrm{Hom}_R(E_{k+1}, E_{k+1})$ to some $f_{k+1} = \mathrm{Hom}_R(P, P_{k+1})$ as follows:

$$\begin{array}{cccccccc} f_{k+1} & \longrightarrow & f_k\psi_0 & & & & & \\ & & \uparrow & & & & & \\ & & f_k & \longrightarrow & \cdot & & & \\ & & & & \cdot & & & \\ & & & & \uparrow & & & \\ & & & & f_2 & \longrightarrow & f_1\psi_{k-1} & \\ & & & & & & \uparrow & \\ & & & & & & f_1 & \longrightarrow \psi_k \\ & & & & & & & \uparrow \\ & & & & & & & \iota \end{array}$$

Case 1. $f_i\psi_{k-i} \neq 0$, $i = 1, \cdots, k$. In this case consider

$$
\begin{array}{cccccc}
f_{k+1} & 0 & & & & \\
\uparrow & \uparrow & & & & \\
h_1 & \phi_{k+1}h_1 - f_k & 0 & & & \\
 & \uparrow & \uparrow & & & \\
 & h_2 & \phi_k h_2 - f_{k-1} & \ddots & & \\
 & & \uparrow & & & \\
 & & h_3 & 0 & & \\
 & & & \uparrow & & \\
 & & & \phi_3 h_{k-1} - f_2 & 0 & \\
 & & & \uparrow & \uparrow & \\
 & & & h_k & \phi_2 h_k - f_1 & 0 \\
 & & & & \uparrow & \uparrow \\
 & & & & h_{k+1} & \phi_1 h_{k+1} - \iota
\end{array}
$$

Since $(\phi_1 h_{k+1} - \iota)\psi_k = 0$, $\phi_1 h_{k+1}\psi_k = \psi_k$. But if $\operatorname{Ker}\phi_1 h_{k+1} \neq 0$, then $\operatorname{Im}\psi_k \cap \operatorname{Ker}\phi_1 h_{k+1} \neq 0$, as $\operatorname{Im}\psi_k$ is essential in E_{k+1}. This is impossible, so h_{k+1} is monic. Thus, E_{k+1} is isomorphic to a submodule of P_1 and so, is projective.

Case 2. $f_i\psi_{k-i} = 0$ for some $1 \leq i \leq k$. Let l be the minimal such i. Then we are back in the situation for Case 1 with $k = l - 1$.

($\Leftarrow$) This is the dual of the sufficiency proof.

For those rings for which the dimensions, respectively, of the injectives and left projectives are positive, the theories of codominant dimension and dominant dimension coincide.

Proposition 3.2. *Let* R *be left perfect and* $\operatorname{dom}\dim_R \mathcal{P} \neq 0$. *Then,* $\operatorname{codom}\dim_R \mathfrak{U} = \operatorname{dom}\dim_R R$.

Proof. If not, then by (1.1), (2.1) and (2.4), $\operatorname{codom}\dim_R \mathcal{J} = 0$.

Corollary 3.3. *If* R *is left perfect and* $\operatorname{dom}\dim_R \mathcal{P} \neq 0$, *then* $\operatorname{codom}\dim_R \mathfrak{U} = \operatorname{codom}\dim \mathfrak{U}_R$, *where* $\mathfrak{U}_R$ *is the minimal injective cogenerator for* $\mathfrak{M}_R$.

Proof. As noted in (1.2), R must be semiprimary. But then by [5, Theorem 10], right and left dominant dimensions are equal.

To throw slightly more light on those rings for which $\operatorname{dom}\dim_R \mathcal{P} \neq 0$, we make the following observations.

Proposition 3.4. *Let* R *be left perfect and* $\operatorname{dom}\dim_R R > 0$. *Then, the projective covers of injective left modules are direct sums of finitely generated injectives.*

Proof. For a projective cover $\bigoplus\Sigma Re_\alpha \to {}_RE \to 0$, we note that each Re_α is a submodule of an injective Re_β. Considering

$$\begin{array}{ccccc} 0 & & & & \\ \downarrow & & & & \\ \bigoplus\sum Re_\alpha & \longrightarrow & {}_RE & \longrightarrow & 0 \\ \downarrow & \nearrow & & & \\ \bigoplus\sum Re_\beta & & & & \end{array}$$

we note that $\bigoplus\Sigma Re_\alpha$ is a direct summand of $\bigoplus\Sigma Re_\beta$ and, hence, of the desired form.

Corollary 3.5. *Let R be left perfect and* dom $\dim_R R > 0$. *Then, if ${}_R\mathfrak{U}$ is finitely generated or if the injective indecomposable projectives are Σ-injective,* codom $\dim_R \mathfrak{U} \neq 0$.

Proposition 3.6. *Let R be perfect and* codom $\dim_R \mathfrak{U} \neq 0$. *Then, R has an injective projective faithful left ideal.*

Proof. Let Re be an indecomposable projective. Then,

$$\begin{array}{ccccc} 0 \longrightarrow Re & \longrightarrow & E(\text{Soc } Re) & \cong & E(Re) \\ & \searrow & \uparrow\phi & & \\ & & \bigoplus\sum\limits_B Re_\beta & & \end{array}$$

where ϕ is the projective cover mapping, can be completed with a monic map. Since Re is finitely generated, $\text{Soc}(Re)$ is isomorphic to a submodule of $\bigoplus\Sigma_{B'}Re_\beta$ for some finite subset $B' \subseteq B$. For a given minimal submodule S of Re, there exists $\beta' \in B'$ such that $S \cong S_{\beta'} \leq Re_{\beta'}$ and $\phi(S_{\beta'}) \cong S_{\beta'}$. Thus, the composition $\rho\phi$: $\bigoplus\Sigma_B Re_\beta \to E(S)$, where ρ: $E(\text{Soc } Re) \to E(S)$ is the natural projection, does not annihilate the canonical image of $S_{\beta'}$. On the other hand, $\bigoplus\Sigma_B Re_\beta = P \oplus P'$, where the restriction of $\rho\phi$ to P provides a projective cover and P' is annihilated by $\rho\phi$. Hence, S is isomorphic to a submodule of P, and so there exists an injective Re_β with S as a submodule. Consequently, R has an injective left ideal which contains a copy of each minimal left ideal of R.

Proposition 3.7. *Let R be left perfect, and* dom $\dim_R R > 0$ *or* codom $\dim_R \mathfrak{U} > 0$. *Then,* dom $\dim_R \mathfrak{P} \neq 0$ *if and only if each injective projective left module is Σ-injective.*

Proof. ($\Rightarrow$) If Re is injective and not Σ-injective, then dom dim $\bigoplus\Sigma_\Omega Re = 0$ for infinite Ω.

($\Leftarrow$) See (2.4).

$$
\begin{array}{ccccccccccccc}
0 & & 0 & & 0 & & & & 0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Hom}_R(P, P_{k+1}) & \to & \mathrm{Hom}_R(P, P_k) & \to & \mathrm{Hom}_R(P, P_{k-1}) & \to & \cdots & \to & \mathrm{Hom}_R(P, P_2) & \to & \mathrm{Hom}_R(P, P_1) & \to & \mathrm{Hom}_R(P, E_{k+1}) & \to 0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Hom}_R(E_1, P_{k+1}) & \to & \mathrm{Hom}_R(E_1, P_k) & \to & \mathrm{Hom}_R(E_1, P_{k-1}) & \to & \cdots & \to & \mathrm{Hom}_R(E_1, P_2) & \to & \mathrm{Hom}_R(E_1, P_1) & \to & \mathrm{Hom}_R(E_1, E_{k+1}) & \to 0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Hom}_R(E_{k-1}, P_{k+1}) & \to & \mathrm{Hom}_R(E_{k-1}, P_k) & \to & \mathrm{Hom}_R(E_{k-1}, P_{k-1}) & \to & \cdots & \to & \mathrm{Hom}_R(E_{k-1}, P_2) & \to & \mathrm{Hom}_R(E_{k-1}, P_1) & \to & \mathrm{Hom}_R(E_{k-1}, E_{k+1}) & \to 0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Hom}_R(E_k, P_{k+1}) & \to & \mathrm{Hom}_R(E_k, P_k) & \to & \mathrm{Hom}_R(E_k, P_{k-1}) & \to & \cdots & \to & \mathrm{Hom}_R(E_k, P_2) & \to & \mathrm{Hom}_R(E_k, P_1) & \to & \mathrm{Hom}_R(E_k, E_{k+1}) & \to 0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Hom}_R(E_{k+1}, P_{k+1}) & \to & \mathrm{Hom}_R(E_{k+1}, P_k) & \to & \mathrm{Hom}_R(E_{k+1}, P_{k-1}) & \to & \cdots & \to & \mathrm{Hom}_R(E_{k+1}, P_2) & \to & \mathrm{Hom}_R(E_{k+1}, P_1) & \to & \mathrm{Hom}_R(E_{k+1}, E_{k+1}) & \to 0
\end{array}
$$

Figure 1

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