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SPHERICAL CURVES AND THEIR ANALOGUES IN AFFINE DIFFERENTIAL GEOMETRY[1]

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ABSTRACT. Necessary and sufficient conditions for curves in Euclidean space to be spherical are derived in a fashion which can be generalized to affine differential geometry and analogues of those curves. This also includes a discussion of some geometrical aspects in recent papers by S. Breuer, D. Gottlieb, and Y.-C. Wong.

1. **Introduction.** The problem of characterizing classes of ordinary differential equations which can be transformed into equations with constant coefficients was recently considered by S. Breuer and D. Gottlieb [1]. In [2] the authors gave an application to spherical curves by deriving a simple differential equation for the radius of curvature. This result was later used by Y.-C. Wong [9] in connection with his criterion in [8] for a curve to be spherical.

We want to show that the formal approach in [2] has an underlying geometrical idea, which yields the reason for the simplicity of the result in [2], and we shall also give a geometrical motivation and simplified derivation of Wong's result.

Furthermore, that idea can be modified so that it becomes applicable in affine differential geometry. This field was developed by E. Salkowski [6] and others. In a more general and abstract form it has recently become important mainly for functional analytic reasons. In fact, such a "differential geometry of vector spaces" was initiated by E. R. Lorch [4] and R. Nevanlinna [5] and has applications in physics, for instance in the theory of elasticity of anisotropic media. The notion of a sphere does not make sense in that geometry, but we shall define spherical curves in affine space in terms of a property which is characteristic of those curves in Euclidean space. We shall also obtain a condition necessary and sufficient for a curve in affine space to be spherical.

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2. Notations. Modified Frenet formulas. Let $x\colon J \to E_3$ represent a curve C in Euclidean space E_3, where $J \subset R$ is any fixed open interval. We always assume that C has a unique tangent and positive curvature κ on J and all appearing derivatives exist and are continuous functions of the arc length s of C on J. We also assume that the torsion τ of C is not zero (but shall drop this assumption later). Let t, p, b be the trihedron of C, and define α, β, γ by

$$d\alpha = \kappa\, ds, \qquad d\beta = \sqrt{\kappa^2 + \tau^2}\, ds, \qquad d\gamma = \tau\, ds; \tag{1}$$

these quantities are usually called the *angles of contingence* of the tangent, principal normal, and binormal, respectively, and γ will play a crucial role in our approach. Setting $\lambda = \kappa/\tau$, we can easily obtain Frenet formulas for derivatives with respect to γ (denoted by dots):

$$\dot{t} = \lambda p, \qquad \dot{p} = b - \lambda t, \qquad \dot{b} = -p. \tag{2}$$

3. Differential equation for ρ. A curve C is spherical iff there is a point in common with all normal planes of C. This holds iff there is a cone S passing through C and such that any generator of S through a point $P \in C$ lies in the normal plane of C at P. Clearly, S is a ruled surface which can be represented in the form

$$y(r, s) = x(s) + r(A(s)p(s) + B(s)b(s)),$$

where x represents C. The surface S is a cone iff there is a function r of s such that

$$x(s) + v(s)p(s) + w(s)b(s) = k, \tag{3}$$

where $v(s) = r(s)A(s)$, $w(s) = r(s)B(s)$, and k is a constant vector. This is equivalent to

$$\dot{x} + \dot{v}p + v\dot{p} + \dot{w}b + w\dot{b} = 0.$$

Applying (2) and $\dot{x} = t/\tau$, we have

$$\tau^{-1}t + \dot{v}p + v(b - \lambda t) + \dot{w}b - wp = 0.$$

Equating the coefficients of t, p, b to zero, we have successively

$$v = \rho, \qquad w = \dot{v} = \dot{\rho}, \qquad v + \dot{w} = 0.$$

The last relation becomes simply

$$\ddot{\rho} + \rho = 0. \tag{4}$$

Solutions are of the form

$$\rho(s) = R\cos(\gamma(s) + \gamma_0), \qquad \gamma(s) = \int_{s_0}^{s} \tau(\tilde{s})\, d\tilde{s}, \tag{5}$$

where R is the radius of the sphere of C. Note that these solutions depend on the torsion. (5) is a necessary and sufficient condition for $C \subset E_3$ to be spherical (even when $\tau = 0$ for some s). (4) and (5) were obtained in [2] in an entirely different way, and our derivation shows that the simplicity of the result is achieved because we used the angle of contingence of the binormal.

4. **Another natural equation.** A more familiar (but more complicated) natural equation of a spherical curve with $\tau \neq 0$ is readily obtained from (4) and (1):

$$(\rho'/\tau)' + \rho\tau = 0, \tag{6}$$

where primes denote derivatives with respect to s. Of course other ways of deriving (6) can be modified so that they yield (4). For instance, assuming the formula for the centers of curvature, we have

$$z = x + \rho p + (\rho'/\tau)b = x + \rho p + \dot{\rho} b.$$

C is spherical iff z is constant; thus

$$\dot{z} = \dot{x} + \dot{\rho} p + \rho\dot{p} + \ddot{\rho} b + \dot{\rho}\dot{b} = 0.$$

From this and (2), equation (4) follows.

5. **On theorems by Y.-C. Wong.** A curve C: $x(s)$ with nonzero curvature κ and the torsion τ is spherical iff (6) holds. If $\tau = 0$ at some s, then (6) is no longer applicable. This case was considered by Y.-C. Wong [8], [9] who proved the following two theorems.

I. *A curve* C: $x \in C^4(J)$, $J = [s_1, s_2]$, *in* E_3 *with a unique tangent is spherical iff it satisfies the two conditions*:

(i) $\kappa(s) > 0$ *in* J;

(ii) *there is a function* $f \in C'(J)$ *such that*

$$f\tau = \rho', \qquad f' = -\rho\tau \qquad (s \in J). \tag{7}$$

II. *The curve* C *in Theorem* I *is spherical iff it satisfies* (5).

Note that it is not difficult to see that a spherical curve with a unique tangent has positive curvature and thus a unique trihedron.

Wong proved II by showing that (5) satisfies (i) and (ii) and, conversely, can be obtained from (i) and (ii). In [9] he expresses surprise that this is so. We shall give another proof of I which rests on geometrical arguments and thus explains the geometrical background of Wong's approach and the connection between I and II.

A sphere in E_3 having contact of second order with a curve C: $x(s)$ has center $a = x + \rho p + hb$ where h is arbitrary; cf. [3, p. 54]. If for every s we associate with C such a sphere, these spheres have constant radius $|a - x|$ iff

$$\rho\rho' + hh' = 0. \tag{8}$$

These spheres have contact of third order iff $\rho\kappa' + h\kappa\tau = 0$; cf. [3, p. 54] which is equivalent to

$$\rho' = h\tau. \tag{9a}$$

From this and (8) we have

$$\rho\tau + h' = 0. \tag{9b}$$

If $\tau \neq 0$, we get $h = \rho'/\tau$, and (9b) yields (6) as a necessary and sufficient condition for C to be spherical. If $\tau = 0$ for some s, equations (9a) and (9b) still make sense and are precisely the condition (7) (with f denoted by h).

Note that we were dealing with osculating spheres, which have contact of third order with the curve, so that Wong's assumption $x \in C^4(J)$ is natural, albeit not the weakest one. Note further that $\tau = 0$ at an s_0 implies $\rho' = 0$ at s_0; cf. (9a); that is, if the osculating plane is stationary ($b' = 0$) and C is spherical, then κ must be stationary at that point. This is geometrically understandable.

To explain the connection between I and II in geometrical terms, we may set $h = \dot{H}$ in (9a) and integrate. Then we see that $\rho = H + c$. Hence the function f in (7) is geometrically the derivative of the radius of curvature with respect to γ. In (9b) we then have

$$\tau(\rho + \ddot{H}) = \tau(\rho + \ddot{\rho}) = 0,$$

which gives a reason for the relation between I and II.

6. **Some concepts of affine differential geometry.** We want to show that the idea of §3 can be generalized to curves in affine space A_3. For this we shall need a few simple concepts and facts as follows. Affine differential geometry investigates invariants with respect to the group of those affine transformations

$$x_j^* = \sum_{k=1}^{3} a_{jk} x_k + c_j \qquad (j = 1, 2, 3)$$

which are volume-preserving ($\det(a_{jk}) = 1$). Such a transformation is said to be *equiaffine*, and affine differential geometry is also known as *equiaffine differential geometry*.

Let $x\colon J \to A_3$ represent a curve C in A_3, where $J = (u_1, u_2) \subset R$ is any fixed open interval. We assume that $x \in C^4(J)$ and $|\dot{x}\ \ddot{x}\ \dddot{x}| \neq 0$ on J, where $\dot{x} = dx/du$, etc. Then with C we may associate the invariant parameter

$$\sigma(u) = \int_{u_1}^{u} |\dot{x}\ \ddot{x}\ \dddot{x}|^{1/6}\, du^* \qquad (\dot{} = d/du^*)$$

which is called the *affine arc length* of C and yields a representation $x(\sigma)$ of C. A *trihedron* of C consists of the *tangent vector* $t = x'$, the *affine normal vector* $p = x''$, and the *affine binormal vector* $b = x'''$; here $x' = dx/d\sigma$, etc. The vectors p and b span the *affine normal plane*. The *affine Frenet formulas* are

$$t' = p, \qquad p' = b, \qquad b' = -\tilde{\tau}t - \tilde{\kappa}p. \tag{10}$$

They involve the invariants

$$\tilde{\kappa} = |x'\ x'''\ x^{\mathrm{iv}}|, \qquad \tilde{\tau} = -|x''\ x'''\ x^{\mathrm{iv}}|,$$

which are called the *affine curvature* and *torsion* of C.

7. **Spherical curves in affine space A_3.** The property of spherical curves in E_3 stated at the beginning of §3 suggests the following

Definition. A curve $C \subset A_3$ is said to be *spherical* if all affine normal planes of C pass through a common point in A_3.

As in §3 this holds iff there is a cone $S \supset C$ such that any generator of S through a point $P \in C$ lies in the affine normal plane of C at P. This holds iff there is a function r of σ such that in the representation

$$y(r, \sigma) = x(\sigma) + r(A(\sigma)p(\sigma) + B(\sigma)b(\sigma))$$

we have

$$x(\sigma) + v(\sigma)p(\sigma) + w(\sigma)b(\sigma) = k,$$

where $v(\sigma) = r(\sigma)A(\sigma)$, $w(\sigma) = r(\sigma)B(\sigma)$, and k is a constant vector. This is equivalent to

$$t + v'p + vp' + w'b + wb' = 0.$$

Applying (10) and equating to zero the coefficients of the independent vectors t, p, b, we have the three conditions

$$\tilde{\tau} w = 1, \qquad v' = \tilde{\kappa} w, \qquad v + w' = 0.$$

A solution is

$$\tilde{\kappa} = \tilde{\tau}(\tilde{\tau}'/\tilde{\tau}^2)'. \tag{11}$$

Performing the indicated differentiation, we could cast this in the form of a Riccati equation, which also appeared in a paper by L. A. Santalo [7] who obtained it in a different way. However, (11) and its derivation, together with suitable differentiability assumptions, give immediately the following remarkable criterion.

Theorem. *Let C: $x(u)$ be a curve in affine space A_3 which is of class $C^6(J)$ on some fixed open interval J and satisfies $|\dot{x}\ \ddot{x}\ \dddot{x}| \neq 0$ and $\tilde{\tau} \neq 0$ on J, where $\tilde{\tau}$ is the affine torsion of C. Then C is spherical (definition above) iff*

$$\chi'' + \tilde{\kappa}(\sigma)\chi = 0, \tag{12}$$

where σ and $\tilde{\kappa}$ are the affine arc length and affine curvature of C, and $\chi = 1/\tilde{\tau}$ $(\tilde{\tau} \neq 0)$ is the affine radius of torsion of C.

A geometrical discussion of (12) will be presented at some other occasion.

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