PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 2, April 1975

AN EXTENDED INEQUALITY FOR THE MAXIMAL FUNCTION

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ABSTRACT. Fefferman and Stein [3] have proved an L^p inequality for the Hardy-Littlewood maximal function applied to functions taking values in a sequence space l^p. This note extends their theorem to functions taking values in a mixed L^p space. An application to mixed estimates for Riesz potentials is given.

1. **Notation.** Let $k = (k_1, \cdots, k_n)$ be an n-tuple of natural numbers, and let $P = (p_1, \cdots, p_n)$, where $1 \le p_i < \infty$. For $f = \{f_k\}_{k \in N^n}$, $K_P(f)$ is defined by successively computing the l^{p_i} norm with respect to k_i while $k_{i+1}, \cdots, k_n$ are held fixed. The set of sequences on N^n for which this norm is finite is denoted by l^P.

For ϕ a complex-valued locally integrable function on R^m, ϕ^* denotes the maximal function

$$\phi^*(x) = \sup_{r>0} \frac{1}{mB(x, r)} \int_{B(x,r)} |\phi(y)|\, dy.$$

Here $B(x, r)$ is the ball with center x and radius r; $mB(x, r)$ is its measure.

For $f = \{f_k\}$ a function on R^m with values in l^P, f^* is the l^P-valued function obtained by taking the maximal function of each f_k.

2. **Theorem.** *For $1 < p_i < \infty$ and $1 < q < \infty$, there is a constant c such that*

$$\int K_P(f^*)^q\, dx \le c \int K_P(f)^q\, dx.$$

Proof. We perform induction on n. Fefferman and Stein [3] have treated the case $n = 1$; we could just as well start with $n = 0$. Assume the Theorem has been established for some fixed value of n. We show that for $f = \{f_{j,k}\}_{j \in N, k \in N^n}$ and $1 < r < \infty$,

Received by the editors January 25, 1974.
AMS (MOS) subject classifications (1970). Primary 46E30, 26A86.
Key words and phrases. Maximal function, mixed L^p norm.

$$\int K_P\left[\left(\sum_{j=1}^{\infty}|f^*_{j,k}|^r\right)^{1/r}\right]^q dx \le c\int K_P\left[\left(\sum_{j=1}^{\infty}|f_{j,k}|^r\right)^{1/r}\right]^q dx$$

or, more briefly,

$$(*)\qquad \int K_P J_r(f^*)^q\,dx \le c\int K_P J_r(f)^q\,dx.$$

Note that for each j, k we have

$$|f_{j,k}(x)| \le \sup_j |f_{j,k}(x)| = J_\infty(f_{\cdot,k})(x).$$

Thus $f^*_{j,k}(x) \le J_\infty(f_{\cdot,k})^*(x)$, and so $J_\infty(f^*_{\cdot,k}) \le J_\infty(f_{\cdot,k})^*$. Hence

$$\int K_P J_\infty(f^*)^q\,dx \le \int K_P[J_\infty(f)^*]^q\,dx \le c\int K_P[J_\infty(f)]^q\,dx$$

by the inductive hypothesis. Thus $(*)$ is valid for $r=\infty$.

Now suppose $1 < r < \min(q, p_1, \cdots, p_n)$. Note that

$$K_P J_r(f^*) = K_P\left[\left(\sum_{j=1}^{\infty}|f^*_{j,\cdot}|^r\right)^{1/r}\right] = K_{P/r}\left[\sum_{j=1}^{\infty}|f^*_{j,\cdot}|^r\right]^{1/r},$$

where $P/r = (p_1/r, p_2/r, \cdots, p_n/r)$. Thus by the duality established in Benedek and Panzone [2],

$$\int K_P J_r(f^*)^q\,dx = \int K_{P/r}\left[\sum_{j=1}^{\infty}|f^*_{j,\cdot}|^r\right]^{q/r} dx = \sup_\phi\left|\int\sum_k\left[\sum_{j=1}^{\infty}|f^*_{j,k}|^r\right]\phi_k\,dx\right|^{q/r},$$

where the supremum is taken over all $\phi=\{\phi_k\}$ for which $\int K_{(P/r)'}(\phi)^{q/(q-r)}dx \le 1$. By Lemma 1 of [3],

$$\begin{aligned}\left|\int\sum_k\left[\sum_{j=1}^{\infty}|f^*_{j,k}|^r\right]\phi_k\,dx\right| &\le \sum_k\sum_{j=1}^{\infty}\int|f^*_{j,k}|^r|\phi_k|\,dx \le c\sum_k\sum_{j=1}^{\infty}\int|f_{j,k}|^r\phi^*_k\,dx\\ &\le \int K_{P/r}\left[\sum_{j=1}^{\infty}|f_{j,\cdot}|^r\right]K_{(P/r)'}[\phi^*]\,dx\\ &\le \left(\int K_{P/r}\left[\sum_{j=1}^{\infty}|f_{j,}\ |^r\right]^{q/r}dx\right)^{r/q}\\ &\quad\cdot\left(\int K_{(P/r)'}[\phi^*]^{q/(q-r)}dx\right)^{1-r/q}\end{aligned}$$

using successive applications of Hölder's inequality for sequences and integrals. By the inductive hypothesis

$$\int K_{(P/r)'}[\phi^*]^{q/(q-r)}\,dx \le c\int K_{(P/r)'}[\phi]^{q/(q-r)}\,dx \le c.$$

Since $\int K_{P/r}[\Sigma_{j=1}^{\infty}|f_{j,\cdot}|^r]^{q/r}\,dx = \int K_P J_r(f)^q\,dx$, this yields (*).

We complete the proof by an interpolation process. Benedek and Panzone [2] give an extension of the Riesz-Thorin interpolation theorem to mixed L^p spaces. However, the Theorem applies directly only to linear operators, and the operator under consideration is nonlinear.

Instead we look at a linear operator T defined by

$$(Tf)_{i,j,k}(x) = \frac{1}{mB(x,\ 2^i)}\int_{B(x,2^i)} f_{j,k}(x)\,dx.$$

Obviously

$$I_\infty[(Tf)_{\cdot,j,k}(x)] = \sup_{-\infty<i<\infty} |(Tf)_{i,j,k}(x)| \le f^*_{j,k}(x)$$

while $f^*_{j,k}(x) \le 2^m I_\infty[(T|f|)_{\cdot,j,k}(x)]$. (Here $|f| = \{|f_{j,k}|\}$.) Thus, (*) holds for all f if and only if

$$(**)\qquad \int K_P J_r I_\infty(Tf)^q\,dx \le c\int K_P J_r(f)^q\,dx$$

for all f.

We have seen that (*) and, hence, (**) is valid for $r = \infty$ and for $1 < r < \min(q, p_1, \cdots, p_n)$; interpolation yields (**) for $1 < r \le \infty$. Thus, (*) is valid for $1 < r \le \infty$ and the Theorem is proved.

3. **A generalization.** There is no particular difficulty in replacing the discrete variable k by a continuous variable t. Let (Ω_i, μ_i) be σ-finite measure spaces, and let $t = (t_1, \cdots, t_n) \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n = \Omega$. For $f(x, t)$ a locally integrable function on $R^m \times \Omega$, let

$$f^*(x,\ t) = \sup_{r>0} \frac{1}{mB(x,\ r)}\int_{B(x,r)} |f(y,\ t)|\,dy.$$

Then we have $\int T_P(f^*)^q\,dx \le c\int T_P(f)^q\,dx$, where T_P denotes the mixed L^p norm taken with respect to t.

4. **An application.** In [1], mixed norm estimates are obtained for Riesz potentials. One technique used there involves estimates in terms of a maximal function taken with respect to one of the variables. Then the proof is completed via the Calderón-Zygmund theory of singular integrals. An

alternate approach would be to use the Theorem stated above. This method would improve the exponent in Theorem 3 and remove the restriction $p_{l+1} \geq p_{l+2} \geq \cdots \geq p_n$ in Theorem 3' of [1].

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