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LEMMA ON MEASURABLE CARDINALS

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ABSTRACT. An ordinal is moved by only finitely many measurable cardinals.

We inductively define a class M, analogously to L, Gödel's class of constructible sets. Let $M_0 = \emptyset$; $M_\lambda = \bigcup\{M_\alpha : \alpha < \lambda\}$; and let $M_{\alpha+1}$ be the collection of subsets of M_α definable using parameters by some formula of the infinitary language $L_{\omega_1\omega_1}$ (instead of $L_{\omega\omega}$, the usual finitary language). Let $M = \bigcup\{M_\alpha : \alpha \in \mathrm{OR}\}$; Chang [1] called this class C_{ω_1}. Kunen [2] used the lemma below to show that if there are ω_1 measurable cardinals, then the axiom of choice is false in M. Kunen's proof used the machinery of iterated ultrapowers. The present proof can be read by anyone familiar with the fundamental paper of Scott [3].

A free κ-complete ultrafilter U on κ induces an elementary embedding j of V into the transitive realization N of the ultrapower V^κ/U. We say κ moves δ when there is a U on κ so that $j(\delta) \neq \delta$.

Lemma. *An ordinal is moved by only finitely many measurable cardinals.*

Proof. Suppose $j(\delta) \neq \delta$. Then there is a maximal interval of ordinals moved by j containing δ. It is straightforward to show that this interval is of the form $[\alpha, \beta)$ where $\mathrm{cf}(\alpha) = \kappa$, and $\beta = \sup j^n(\gamma)$ for $\gamma \in [\alpha, \beta)$.

Claim 1. If $\kappa_0 < \kappa_1$ then $j_1(j_0(\gamma)) = j_0(j_1(\gamma))$.

Informally, the idea is that $j_1(j_0(\gamma))$ is N_1's notion of the j_0 process applied to $j_1(\gamma)$ (N_1's notion of γ), which is $j_0(j_1(\gamma))$ because N_1's notion is correct.

Let $\phi_i(x, y, U_i)$ be a formula of the language of ZF, obtained in the familiar manner, such that for all x, y, $\phi_i(x, y, U_i)$ iff $j_i(x) = y$ $(i = 0, 1)$.

Then for every $\gamma \in \mathrm{OR}$, $\phi_0(\gamma, j_0(\gamma), U_0)$. Clearly

$$\phi_0(\gamma, j_0(\gamma), U_0) \Leftrightarrow \phi_0^{j_1''V}(j_1(\gamma), j_1(j_0(\gamma)), j_1(U_0))$$

$$\Leftrightarrow \phi_0^{N_1}(j_1(\gamma), j_1(j_0(\gamma)), U_0) \Leftrightarrow \phi_0(j_1(\gamma), j_1(j_0(\gamma)), U_0).$$

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Since $j_0(j_1(\gamma))$ is the unique y such that $\phi_0(j_1(\gamma), y, U_0)$, we get from $\phi_0(j_1(\gamma), j_1(j_0(\gamma)), U_0)$

$$j_1(j_0(\gamma)) = j_0(j_1(\gamma)).$$

Claim 2. If $[\alpha_0, \beta_0)$ and $[\alpha_1, \beta_1)$ intersect, they are nested. Suppose $\alpha_0 < \alpha_1$. Let $\alpha_0 < \gamma < \alpha_1$, $j_1(\gamma) = \gamma$. Then $j_1(j_0^n(\gamma)) = j_0^n(\gamma)$, so no $j_0^n(\gamma) \in [\alpha_1, \beta_1)$. Then either $\beta_0 = \sup j^n(\gamma) < \alpha_1$ or there is n so that $j_0^n(\gamma) < \alpha_1 < \beta_1 < j_0^{n+1}(\gamma)$.

Let δ, $\{\kappa_i : i \in \omega\}$ be a counterexample to the Lemma. Since $\mathrm{cf}(\alpha_i) \neq \mathrm{cf}(\alpha_j)$ there is an infinite ascending chain $\{\alpha_i : i \in \omega\}$ of ordinals. By Claim 2, there is an infinite descending chain of ordinals. Contradiction.

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