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INEQUALITIES CONCERNING THE CHARACTERS OF A FINITE GROUP

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ABSTRACT. Given a finite group we provide explicit bounds (in terms of the group order and numbers of conjugacy classes and involutions) for (a) the number of real valued characters of type R; (b) the sum of the degrees of the irreducible characters; (c) the sum of the entries of the character table; (d) the sums (b), (c) restricted to real valued characters. We also provide a bound on the number of elements of order $2n$ in terms of the number of elements of order n.

Let us first observe the following inequalities, which follow by application of the Cauchy-Schwarz inequality if one regards the summations on the left as inner products:

$$\sum a_\chi \leq c^{1/2}\left(\sum n_i\right)^{1/2}, \tag{1}$$

$$\sum_{\chi \text{ real}} a_\chi \leq k_1^{1/2}\left(\sum n_i\right)^{1/2}, \tag{2}$$

$$\left|\sum \chi(a)\right| \leq c^{1/2}\,|C(a)|^{1/2}, \tag{3}$$

$$\left|\sum_{\chi \text{ real}} \chi(a)\right| \leq k_1^{1/2}|C(a)|^{1/2}, \tag{4}$$

$$\sqrt{a} \leq k_1^{1/2}|C(a)|^{1/2}. \tag{5}$$

Here $\sqrt{a} = \Sigma\epsilon(\chi)\chi(a)$ is the number of solutions in G of $y^2 = a$; $\epsilon(\chi) = 0, \pm 1$, depending on whether the irreducible character χ is of type C, R, H (Frobenius-Schur [4]); a_χ is the sum of the elements in the χth row of the character table; $|C(a)| = \Sigma a_\chi \chi(a)$ is the order of the centralizer of $a \in G$ [5]; c is the number of conjugacy classes; k_1 is the number of real conjugacy

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classes; $n_i = |C(a_i)|$ where a_i is an element of the ith class ($a_0 = e$, $n_0 = g = |G|$); $m_1 = \sqrt{e}$; l_1, $l_2 = k_1 - l_1$ are the number of characters of type **R**, **H** respectively; $d = \Sigma\chi(1)$; $d_1 = \Sigma_{\chi \text{ real}}\chi(1)$.

Observe that (4) and (5) both generalize a result of Brauer-Fowler, who prove a slightly stronger result when $a = e$ [2, Theorem 2J], although cf. Remark 1 below; and (1) and (2) frequently provide better upper bounds on the character sums than those of [3], [5].

All of these inequalities are contained in the following statement:

The real symmetric matrix

$$M_a = \begin{bmatrix} \sum n_i & |C(a)| & \sum \epsilon(\chi)a_\chi & \sum\limits_{\chi \text{ real}} a_\chi & \sum a_\chi \\ |C(a)| & |C(a)| & \sqrt{a} & \sum\limits_{\chi \text{ real}} \chi(a) & \sum \chi(a) \\ \sum \epsilon(\chi)a_\chi & \sqrt{a} & k_1 & l_1 - l_2 & l_1 - l_2 \\ \sum\limits_{\chi \text{ real}} a_\chi & \sum\limits_{\chi \text{ real}} \chi(a) & l_1 - l_2 & k_1 & k_1 \\ \sum a_\chi & \sum \chi(a) & l_1 - l_2 & k_1 & c \end{bmatrix}$$

is positive semidefinite.

For (1)–(5) merely state that certain of the 2×2 principal minors obtained by permuting corresponding columns and rows of M_a are nonnegative (there are ten such minors altogether, but the other five lead to trivial or weaker inequalities than those above).

To see that M_a is positive semidefinite, merely observe that $M_a = A_a A_a^*$ where

$$A_a = \begin{bmatrix} \cdots & a_\chi & \cdots \\ \cdots & \chi(a) & \cdots \\ \cdots & \epsilon(\chi) & \cdots \\ \cdots & \epsilon(\chi)^2 & \cdots \\ \cdots & 1 & \cdots \end{bmatrix}$$

is the $5 \times c$ matrix whose χth column is as shown.

To obtain the inequalities which we desire, we permute corresponding rows and columns of M_a and compute the 3×3 principal minors.

We should perhaps mention that we have here a veritable plethora of inequalities: first, for any $a \in G$ (and permutation of corresponding columns and rows) all of the principal minors of M_a are nonnegative; we may also average the M_a's over subsets of G to obtain positive semidefinite matrices; or we may average the principal minors of M_a over subsets of G; or we may use Minkowski's result that the determinant function is concave on the set of positive semidefinite matrices to obtain inequalities between the principal minors of the averaged M_a's and the average of the principal minors of M_a. We will content ourselves here, however, with five (of the ten) 3×3 principal minors of M_a when $a = e$:

$$ck_1g + 2dm_1(l_1 - l_2) - m_1^2c - (l_1 - l_2)^2g - d^2k_1 \geq 0, \tag{6}$$

$$k_1^2g + 2d_1m_1(l_1 - l_2) - m_1^2k_1 - (l_1 - l_2)^2g - d_1^2k_1 \geq 0, \tag{7}$$

$$\left(\sum n_i\right)k_1g + 2m_1g\sum\epsilon(\chi)a_\chi - m_1^2\sum n_i - k_1g^2 - g\left(\sum\epsilon(\chi)a_\chi\right)^2 \geq 0, \tag{8}$$

$$\left(\sum n_i\right)k_1g + 2d_1g\left(\sum_{\chi\text{ real}} a_\chi\right) - d_1^2\sum n_i - k_1g^2 - g\left(\sum_{\chi\text{ real}} a_\chi\right)^2 \geq 0, \tag{9}$$

$$\left(\sum n_i\right)cg + 2dg\left(\sum a_\chi\right) - d^2\sum n_i - cg^2 - g\left(\sum a_\chi\right)^2 \geq 0. \tag{10}$$

From (6), (7) we obtain (after substituting $l_2 = k_1 - l_1$)

Theorem 1.

(A)
$$\frac{k_1}{2} + \frac{m_1d_1}{2g} - \frac{1}{2g}\sqrt{(gk_1 - d_1^2)(gk_1 - m_1^2)} \leq l_1 \leq \frac{k_1}{2} + \frac{m_1d_1}{2g} + \frac{1}{2g}\sqrt{(gk_1 - d_1^2)(gk_1 - m_1^2)},$$

(A′)
$$\frac{k_1}{2} + \frac{m_1d}{2g} - \frac{1}{2g}\sqrt{(gc - d^2)(gk_1 - m_1^2)} \leq l_1 \leq \frac{k_1}{2} + \frac{m_1d}{2g} + \frac{1}{2g}\sqrt{(gc - d^2)(gk_1 - m_1^2)},$$

(B)
$$d_1 \leq m_1\frac{l_1 - l_2}{k_1} + \sqrt{(gk_1 - m_1^2)(k_1^2 - (l_1 - l_2)^2)},$$

(C)
$$d \leq m_1\frac{l_1 - l_2}{k_1} + \sqrt{(gk_1 - m_1^2)(ck_1 - (l_1 - l_2)^2)}.$$

From (8), (9), (10), we obtain

Theorem 2.

$$\sum \epsilon(\chi)a_\chi \le m_1 + \Big(\sum_{i\neq 0} n_i\Big)^{1/2} \sqrt{k_1 - m_1^2/g}, \tag{D}$$

$$\Big|\sum_{\chi \text{ real}} a_\chi - d_1\Big| \le \Big(\sum_{i\neq 0} n_i\Big)^{1/2} \sqrt{k_1 - d_1^2/g}, \tag{E}$$

$$\Big|\sum a_\chi - d\Big| \le \Big(\sum_{i\neq 0} n_i\Big)^{1/2} \sqrt{c - d^2/g}. \tag{F}$$

We conclude with three remarks and an application:

1. Concerning (A) and (A′), one may estimate d_1, d by $m_1 \le d_1 \le k_1^{1/2}g^{1/2}$, $g^{1/2} \le d \le c^{1/2}g^{1/2}$ to obtain bounds on l_1, l_2 strictly in terms of g, m_1, k_1, c. In particular we have $m_1^2/g \le l_1 \le k_1$ which sharpens Brauer-Fowler [2, Theorem 2J] in another direction. We conjecture that $l_1 \ge k_1/2$; determining the exact relationship between l_1, l_2 and the internal structure G is an old unsolved problem [1]. The upper bound in (A) is attained whenever $k_1 = l_1$ (observe $m_1 = d_1$ in this case); the upper bound in (A′) is attained whenever $c = k_1 = l_1$. Note, however, that the upper bounds are also attained for the quaternion group of order 8 (where $l_1 = k_1 - 1$).

2. Concerning (B) and (C), recall $m_1 \le d_1 \le d$ and, as in (A) and (A′), equality holds whenever $k_1 = l_1$ (for (B)) or $c = k_1 = l_1$ (for (C)). One may use the trivial estimate $|l_1 - l_2| \le k_1$ to obtain bounds strictly in terms of g, m_1, k_1, c.

3. Concerning (D), (E) and (F), recall that in [3] we proved that $\Sigma\epsilon(\chi)a_\chi \ge (m_1 - 1) + (c - r) + (k_1 - k_2)$ so $\Sigma a_\chi \ge \Sigma_{\chi\,\text{real}} a_\chi \ge \Sigma\epsilon(\chi)a_\chi \ge m_1$. We conjecture that in fact $\Sigma a_\chi \ge d$ and $\Sigma_{\chi\,\text{real}} a_\chi \ge d_1$. Observe that $\Sigma a_\chi - d$ is the sum of the elements of the character table outside of the first column; and $\Sigma_{\chi\,\text{real}} a_\chi - d_1$ is the sum of such elements in rows corresponding to real valued characters.

An application. Let us again consider the positive semidefinite matrix

$$\begin{bmatrix} k_1 & \sqrt{a} \\ \sqrt{a} & |C(a)| \end{bmatrix}.$$

If we sum these matrices over the set of involutions of G we obtain the positive semidefinite matrix

$$\begin{bmatrix} mk_1 & m_4 \\ m_4 & gr \end{bmatrix}$$

where m denotes the number of involutions, m_4 the number of elements of order 4, and r the number of conjugacy classes of involutions. Hence $m_4 \leq \sqrt{gk_1 mr}$. More generally,

Theorem 3. *If m_n denotes the number of elements of order n and r_n the number of conjugacy classes of elements of order n, then*

$$m_{2n} \leq \sqrt{gk_1 m_n r_n} \qquad \text{if } n \text{ is even},$$

$$m_{2n} \leq \sqrt{gk_1 m_n r_n} - m_n \qquad \text{if } n \text{ is odd.}$$

When $n = 1$ we have again the aforementioned result of Brauer-Fowler that $m \leq \sqrt{gk_1} - 1$. Observe also that for the quaternion group of order 8, $m_4 = 6$ and $gk_1 m_2 r_2 = 40$.

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