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A WEIGHTED NORM INEQUALITY FOR VILENKIN-FOURIER SERIES

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ABSTRACT. Various operators related to the Hardy-Littlewood maximal function have been shown to satisfy a strong type (p, p) condition, $1<p<\infty$, for weighted L^p spaces providing the weight function satisfies the Ap condition of B. Muckenhoupt. In particular this result for the maximal partial sum operator for trigonometric series was established by R. Hunt and W. S. Young. In this note a result similar to that of Hunt and Young is established for Vilenkin-Fourier series, which include Walsh series as a special case.

In a recent paper [5], R. Hunt and W. Young established that the maximal partial sum operator for the trigonometric system is a bounded operator on the weighted L^p spaces, $p > 1$, providing the weight function satisfies the Ap condition introduced by B. Muckenhoupt [6]. In this note we establish a similar result for the Vilenkin systems which include the Walsh system as a special case. This proof follows closely that in [5] but avoids several technical problems encountered in the trigonometric system due to the discrete nature of the underlying group G. In particular, our proof is based on a joint distribution inequality similar to those in [1] and [2].

We assume the reader is familiar with the description and notation of Vilenkin systems (G, X) as discussed in [4]. In particular, we still require that X have a bounded subgroup structure. In the context of Vilenkin systems, a nonnegative weight function $v(x)$ satisfies the Ap condition, $p > 1$, if there exists a constant B such that

$$\left(\mu(\omega)^{-1} \int_\omega v(x)d\mu(x)\right)\left(\mu(\omega)^{-1} \int_\omega v(x)^{-1/(p-1)}d\mu(x)\right)^{p-1} \leq B$$

for all cosets ω of the fundamental sequence of subgroups $\{G_n\}$. Using [3], it is easy to check that each of the following consequences of the Ap condition remained valid in the context of Vilenkin systems:

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(i) If $v(x)$ satisfies the Ap condition with $p > 1$, then there exists r, $1 < r < p$, such that $v(x)$ satisfies the $A_{(p/r)}$ condition.

(ii) Given a measurable set E and a coset ω with $\mu(E \cap \omega) \leq \epsilon\, \mu(\omega)$, there exist positive constants C and δ, independent of E and ω, such that $\mu_v(E \cap \omega) \leq C\epsilon^{\delta}\mu_v(\omega)$ where $\mu_v(F) = \int_F v(x)\,d\mu(x)$ for any measurable set F.

(iii) Let $Hf(x) = \sup_{x \in \omega}(\mu(\omega)^{-1}\int_\omega |f(t)|\,d\mu(t))$ for any integrable f. Then if $f \in L^p(G)$, $p > 1$, and $v(x)$ satisfies the Ap condition, there exists a constant Cp independent of f such that

$$\int_G Hf(x)^p v(x)\,d\mu(x) \leq C_p^p \int_G |f(x)|^p\,d\mu(x).$$

It should be pointed out that the validity of these consequences depends upon the bounded subgroup structure of X.

Let $S_n f(x)$ denote the nth partial sum of the Vilenkin-Fourier series of an integrable f, and let $Mf(x) = \sup_n |S_n f(x)|$. We wish to show that if $v(x)$ satisfies the Ap condition, $p > 1$, then $f \in L^p(G)$ implies

$$\int_G Mf(x)^p v(x)\,d\mu(x) \leq C_p^p \int_G |f(x)|^p v(x)\,d\mu(x)$$

with C_p independent of f. Following [4] we replace $S_n f(x)$ by the modified nth partial sum operator

$$S_n^* f(x) = \chi_n(x) S_n(f\bar{\chi}_n)(x) = (f * D_n^*)(x),$$

where D_n^* denotes the modified nth Dirichlet kernel and $*$ denotes convolution over G. Setting $M^*f(x) = \sup_n |S_n^* f(x)|$, we will show that

$$(1)\qquad \int_G (M^*f(x))^p v(x)\,d\mu(x) \leq C_p^p \int_G |f(x)|^p v(x)\,d\mu(x).$$

For $p > 1$, let

$$H_p f(x) = \sup_{x \in \omega}\left(\mu(\omega)^{-1} \int_\omega |f(t)|^p\,d\mu(t)\right)^{1/p} = (H_1|f|^p(x))^{1/p}.$$

Following Hunt and Young, we will establish the distributional inequality

$$(2)\qquad \mu_v\{x \in G\colon M^* f(x) > 4\lambda,\ H_r f(x) \leq \gamma\lambda\} \leq C(\gamma)\mu_v\{x \in G\colon M^*f(x) > \lambda\}$$

for $\gamma \leq \gamma_0$, where $C(\gamma) \to 0$ as $\gamma \to 0$ and r is given in the first consequence of the Ap condition. To see that (2) implies (1), the reader is referred to [5].

To establish (2) we first note that the set $\{x \in G\colon M^*f(x) > \lambda\}$ is a countable union of disjoint cosets ω_j. This follows from the fact that if $m_k <$

$n \leq m_{k+1}$, $S_n^* f(x)$ is constant on cosets of G_{k+1}. Furthermore, we may assume each ω_j is maximal in the following sense: If $\omega_j = x + G_k$, then there exists a point $z_j \in \omega_j^* = x + G_{k-1}$ such that $M^* f(z_j) \leq \lambda$. Thus it suffices to prove that for each ω_j

$$\mu_v\{x \in \omega_j: M^* f(x) > 4\lambda,\ H_r f(x) \leq \gamma\lambda\} \leq C(\gamma)\mu_v(\omega_j) \tag{3}$$

for $\gamma \leq \gamma_0$. Let $f = f I_{\omega_j^*}(x) + f I_{G \setminus \omega_j^*}(x) \equiv f_1(x) + f_2(x)$ where I_E denotes the characteristic function of a measurable set E. We recall from [4] that for $p > 1$

$$\int_G (M^* f(x))^p d\mu(x) \leq C_p^p \int_G |f(x)|^p d\mu(x). \tag{4}$$

We may assume there exists $u_j \in \omega_j$ with $H_r f(u_j) \leq \gamma\lambda$. Then

$$\begin{aligned} \mu\{x \in \omega_j: M^* f_1(x) > \lambda\} &\leq \lambda^{-r} \int_G (M^* f_1(x))^r d\mu(x) \\ &\leq C_r^r \lambda^{-r} \int_{\omega_j^*} |f_1(x)|^r d\mu(x) \leq C_r^r \lambda^{-r} \mu(\omega_j^*)(H_r f_1(u_j))^r \leq A C_r^r \gamma^r \mu(\omega_j) \end{aligned} \tag{5}$$

since $\mu(\omega_j^*) \leq A\mu(\omega_j)$ follows immediately from the assumption on the subgroup structure of X. We now show that

$$\{x \in \omega_j: M^* f(x) > 4\lambda,\ H_r f(x) \leq \gamma\lambda\} \subset \{x \in \omega_j: M^* f_1(x) > \lambda\}$$

for $\gamma \leq \gamma_0$. Recall from [4] that if $n = \sum_{s=0}^{\infty} a_s m_s$,

$$D_n^*(x) = \sum_{s=0}^{\infty} D_{m_s}(x) \chi_{m_s}^{-a_s} \left(\sum_{j=0}^{a_s - 1} \chi_{m_s}^j(x) \right) \equiv \sum_{s=0}^{\infty} \Phi_{m_s, a_s}(x)$$

where $D_{m_s}(x) = m_s I_{G_s}(x)$. For $x \in \omega_j$ and any $n = \sum_{s=0}^{\infty} a_s m_s$, $S_n^* f_2(x) = S_n^* f_2(z_j)$. To see this let $n_1(\omega_j) = \sum_{s=0}^{k-2} a_s m_s$ and $n_2(\omega_j) = n - n_1(\omega_j)$. Then

$$S_n^* f_2(x) = S_{n_1(\omega_j)}^* f_2(x) + S_{n_2(\omega_j)}^* f_2(x) = (f_2 * D_{n_1(\omega_j)}^*)(x) + (f_2 * D_{n_2(\omega_j)}^*)(x).$$

Now $f_2 * D_{n_2(\omega_j)}^* = 0$ since $D_{n_2(\omega_j)}^*(x)$ has support in G_{k-1}. Also $f_2 * D_{n_1(\omega_j)}^*$ is constant on cosets of G_{k-1} and in particular on ω_j. Thus $S_n^* f_2(x) = S_n^* f_2(z_j)$. Now for $x \in \omega_j$ and any n,

$$
\begin{aligned}
|S_n^* f_2(x) - S_n^* f(z_j)| &= |S_n^* f_1(z_j)| = |(f_1 * D_n^*)(z_j)| \\
&= \left| \int_{\omega_j^*} f_1(t) \sum_{s=0}^{\infty} \Phi_{m_s, a_s}(z_j - t)\, d\mu(t) \right| \\
&\leq \left| \int_{\omega_j^*} f_1(t) \sum_{s=0}^{k-2} \Phi_{m_s, a_s}(z_j - t)\, d\mu(t) \right| \\
&\quad + \left| \int_{\omega_j^*} f_1(t) \sum_{s=k-1}^{\infty} \Phi_{m_s, a_s}(z_j - t)\, d\mu(t) \right| \\
&\leq \int_{\omega_j^*} |f(t)| \left(\sum_{s=0}^{k-2} a_s m_s \right) d\mu(t) + |S^*_{n_2(\omega_j)} f(z_j)| \\
&\leq m_{k-1} \int_{\omega_j^*} |f(t)|\, d\mu(t) + M^* f(z_j) \\
&\leq \mu(\omega_j^*)^{-1} \int_{\omega_j^*} |f(t)|\, d\mu(t) + M^* f(z_j) \\
&\leq H_1 f(u_j) + M^* f(z_j) \leq (\gamma + 1)\lambda.
\end{aligned}
$$

It now follows that $M^* f_2(x) \leq M^* f(z_j) + \lambda(\gamma + 1) \leq \lambda(\gamma + 2)$. Thus for $x \in \omega_j$,

$$M^* f(x) \leq M^* f_1(x) + M^* f_2(x) \leq M^* f_1(x) + \lambda(2 + \gamma).$$

Hence if $M^* f(x) > 4\lambda$, it follows that $M^* f_1(x) > \lambda$ if $\gamma \leq \gamma_0 < 1$. Then for $\gamma \leq \gamma_0$,

$$\mu\{x \in \omega_j : M^* f(x) > 4\lambda,\ H_r f(x) \leq \gamma\lambda\} \leq \mu\{x \in \omega_j : M^* f_1(x) > \lambda\} \leq C\gamma^r \mu(\omega_j)$$

by (5). Using the second consequence of the Ap condition, we obtain

$$\mu_v\{x \in \omega_j : M^* f(x) > 4\lambda,\ H_r f(x) \leq \gamma\lambda\} \leq C(\gamma^r)^\delta \mu_v(\omega_j).$$

Thus (2) is established and the proof is complete.

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