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A NOTE ON JONES' FUNCTION K

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ABSTRACT. For each point x of a continuum M, F. B. Jones [5, Theorem 2] defines $K(x)$ to be the closed set consisting of all points y of M such that M is not aposyndetic at x with respect to y. Suppose M is a plane continuum and for any positive real number ϵ there are at most a finite number of complementary domains of M of diameter greater than ϵ. In this paper it is proved that for each point x of M, the set $K(x)$ is connected.

A continuum M (nondegenerate metric space that is compact and connected) is said to be *aposyndetic* at a point p of M with respect to a point q of M if there exist an open set W and a continuum H in M such that $p \in W \subset H \subset M - \{q\}$.

Throughout this paper S is the set of points of a simple closed surface (2-sphere).

Definition. Let M be a continuum in S and let x and y be distinct points of M. The set $S - M$ is said to be *folded* around x with respect to y if there exist two monotone descending sequences of circular regions $U_1, U_2, U_3, \cdots$ and $V_1, V_2, V_3, \cdots$ in S centered on and converging to x and y respectively such that $\mathrm{Cl}\, U_1 \cap \mathrm{Cl}\, V_1 = \varnothing$ ($\mathrm{Cl}\, U_1$ is the closure of U_1), and there exists a sequence of mutually exclusive sets $X_1, X_2, X_3, \cdots$ in $S - M$ having the following properties. For each positive integer i, the set X_i is the union of two intersecting arc-segments (open arcs) I_i and T_i such that

(1) $I_i \cap T_i$ is connected,

(2) I_i is contained in $\mathrm{Bd}\, U_i$ ($\mathrm{Bd}\, U_i$ is the boundary of U_i) and has endpoints a_i and b_i in M,

(3) the sets $\mathrm{Cl}\, U_{i+1}$ and (a_i-component of $M - V_i$) are disjoint,

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(4) T_i is contained in $S - \mathrm{Cl}(V_i \cup U_{i+1})$ and has two distinct endpoints in $\mathrm{Bd}\, V_i$,

(5) $T_i \cup \mathrm{Bd}\, V_i$ contains a simple closed curve S_i that separates a_i from b_i in S.

Theorem. *If M is a continuum in S and for any positive real number ϵ there are at most a finite number of complementary domains of diameter greater than ϵ, then for each point x of M, the set $K(x)$ is connected.*

Proof. Assume $K(x)$ is not connected. Let y be a point of $K(x)$ that does not belong to the x-component of $K(x)$. There exists an open disk R such that y belongs to R, the disk $\mathrm{Cl}\, R$ is contained in $S - \{x\}$, and M is aposyndetic at x with respect to each point of $M \cap \mathrm{Bd}\, R$ [6, Theorem 49, p. 17 and Theorem 13, p. 170].

Since M is not aposyndetic at x with respect to y, $S - M$ is folded around x with respect to y [4, Theorem 2]. Let $U_1, U_2, U_3, \cdots, V_1, V_2, V_3, \cdots$, and $X_1, X_2, X_3, \cdots$ be the sequences, as described in the definition, which indicate that $S - M$ is folded around x with respect to y. Assume without loss of generality that $\mathrm{Cl}\, U_1 \cap \mathrm{Cl}\, R = \emptyset$ and $\mathrm{Cl}\, V_1 \subset R$.

For each positive integer n, let A_n and B_n denote the a_n-component and the b_n-component of $M - R$ respectively. According to [1, Lemma and the third paragraph in the proof of Theorem 1], we can assume without loss of generality that there exist disjoint arc-segments C and E in $\mathrm{Bd}\, R$ such that for each n, A_n meets both C and E and B_n meets both C and E. For each n, let c_n and e_n be points of $A_n \cap C$ and $A_n \cap E$ respectively. Assume without loss of generality that for each n, A_{n+1} separates A_n from A_{n+2} in $S - R$ [6, Theorem 28, p. 156]. For each n, since the arc-segment I_n is contained in $S - M$, B_{n+1} also separates A_n from A_{n+2} in $S - R$.

The sequence $c_1, c_2, c_3, \cdots$ converges to a point v_1 of $M \cap \mathrm{Cl}\, C$ and $e_1, e_2, e_3, \cdots$ converges to a point v_2 of $M \cap \mathrm{Cl}\, E$. The points v_1 and v_2 are distinct; for otherwise, it would follow that M is not aposyndetic at x with respect to v_1 [1, the fourth paragraph in the proof of Theorem 1].

Since M is aposyndetic at x with respect to each point of $\mathrm{Bd}\, R$, there exist subcontinua H_1 and H_2 of M and circular regions G_1 and G_2 such that $\mathrm{Cl}\, G_1 \cap \mathrm{Cl}\, G_2 = \emptyset$ and such that for $n = 1$ and $n = 2$, the region G_n contains v_n, $H_n \cap \mathrm{Cl}\, G_n = \emptyset$, and the point x is in the interior of H_n relative to M. There is a circular region W that contains x such that $\mathrm{Cl}\, W \cap \mathrm{Cl}\,(G_1 \cup G_2) = \emptyset$ and $W \cap M$ is contained in $H_1 \cap H_2$.

Assume without loss of generality that $\mathrm{Cl}\, C$ is in G_1, $\mathrm{Cl}\, E$ is in G_2, and Cl U_1 is in W. Let $\epsilon = \text{dist}[W, R]$. Since there are at most a finite number of complementary domains of diameter greater than ϵ, there exist integers m and n such that T_m and T_n belong to the same complementary domain of M.

Let T'_m be the component of $T_m - R$ that contains $T_m \cap I_m$ and let T'_n be the component of $T_n - R$ that contains $T_n \cap I_n$. Since $A_m \cup B_m \cup C \cup E$ separates I_m from R in S, we know that T'_m intersects $(G_1 \cup G_2)$. Note also that T'_m intersects both G_1 and G_2, since otherwise the union of T'_m and a component of $\text{Bd}(G_1 \cup G_2)$ would separate a_m from b_m in S [6, Theorem 32, p. 181], and this would contradict the existence of H_1 and H_2. Similarly T'_n intersects both G_1 and G_2.

Since T'_m and T'_n belong to the same complementary domain of M, there is an arc A in $S - M$ that intersects both T'_m and T'_n. Let $K = T'_m \cup T'_n \cup A \cup \text{Bd } G_1$ and let $H = T'_m \cup T'_n \cup A \cup \text{Bd } G_2$. The set $K \cup H$ separates a_m from b_m in S [6, Theorem 32, p. 181]. Since $K \cap H$ is connected, we can assume without loss of generality that K separates a_m from b_m in S [6, Theorem 20, p. 173]. Since H_1 contains $\{a_m, b_m\}$ and misses K, this contradicts the fact that H_1 is a continuum. It follows that $K(x)$ must be connected.

As a consequence of this theorem, we have the following result announced by C. L. Hagopian in [3].

Corollary. *$K(x)$ is connected for each point x of a plane continuum that has only finitely many complementary domains.*

Continua that satisfy the hypothesis of our theorem are called E-continua by G. T. Whyburn. In [7, Theorem 4.4, p. 113] several conditions are given that characterize local connectivity in these spaces. It is proved in [2] that semi-aposyndetic E-continua are arcwise connected.

Example. The set $K(x)$ may fail to be connected for a point x of a plane continuum that is not an E-continuum. To see this, let C be the Cantor discontinuum and define M to be the quotient space

$$C \times [0, 1]/C \times \{0, 1\}.$$

Let y be the separating point of M. Then for each point x of $M - \{y\}$, the set $K(x) = \{x, y\}$.

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