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PRACTICALLY PERFECT THREE-MANIFOLD GROUPS[1]

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ABSTRACT. Let M^3 be a 3-manifold containing no 2-sided projective plane. Let $G \neq \{1\}$ be a finitely-generated subgroup of $\pi_1(M^3)$ such that G is indecomposable relative to free product, and such that G abelianized is finite. (G is "practically perfect".) Then, it is shown that there is a compact 3-submanifold $Z^3 \subset M^3$ such that $\pi_1(Z^3)$ contains a subgroup of finite index conjugate to G, and Z^3 is bounded by a 2-sphere. Some related extensions of this result are given, plus an application to compact absolute neighborhood retracts in 3-manifolds.

1. **Introduction.** Several authors [5], [8], [1] have obtained results concerning conditions under which the fundamental group of a 3-manifold M^3 can have elements of finite order. The best result in this direction is D. B. A. Epstein's Theorem 8.2 of [1], which says that a finite subgroup G of $\pi_1(M^3)$ can arise only in one of two ways: either $G \cong Z_2$ and G "conjugates" into a 2-sided projective plane in M^3, or M^3 has a connected sum factor with finite fundamental group and G conjugates into this factor. Our aim here is to obtain some analogous results in the case that G/G' is finite. (G' denotes the commutator subgroup of G.)

There are some new things to note in our context, however (as contrasted to [1]). First, finiteness of G/G' does *not* prevent G from being a nontrivial free product. (A finite group is indecomposable relative to free product.) Secondly, the nonorientable case is trickier since, even if G/G' is finite and G is indecomposable relative to free product, a subgroup of index two in G may not inherit these properties. The second difficulty seems more serious, and has led us routinely to assume that M^3 contains no 2-sided projective planes.

The basic tools (Theorems 1 and 1$'$) lean heavily on Epstein's results.

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Theorem 2 is our "G/G' finite" analog of his theorem about finite subgroups, and we assume in it that the group G in question is *$*$-indecomposable* (i.e., indecomposable relative to free product, which is denoted by $*$). Theorem 3 is the best we have been able to do when G is not assumed $*$-indecomposable. Finally, Theorem 4 is a geometrically-oriented application of these results to certain compact absolute neighborhood retracts in 3-manifolds. The following result may be easily deduced from either Theorem 2 or Theorem 4, and the reader may wish to take it as an exercise: If X is a compact, connected absolute neighborhood retract in E^3 (E^n denotes Euclidean n-space) and the homology group $H_1(X; Z)$ is finite, then X is simply connected.

All mappings, manifolds, submanifolds, etc. are to be taken in the piecewise-linear sense. A "manifold" is connected. If M^3 is a 3-manifold, then $\hat{M}^3$ is the 3-manifold obtained by attaching a 3-cell to each 2-sphere in its boundary, ∂M^3. A space X is *aspherical* if $\pi_n(X) = 0$ for $n \geq 2$. A 3-manifold M^3 is *irreducible* if each 2-sphere in M^3 bounds a 3-cell in M^3. A good reference for 3-manifold terminology is [7]. We use Δ^n to denote an n-simplex, $I = \Delta^1$, and S^n is $\partial\Delta^{n+1}$. Real projective n-space is P^n. Homeomorphism of spaces A, B is symbolized $A \approx B$; and the fact that groups G, H are isomorphic is symbolized $G \cong H$. The infinite cyclic group is Z; the cyclic group of finite order n is Z_n. Homology groups are singular and have Z coefficients unless otherwise indicated.

2. **Surrounding a subgroup.** We collect in one theorem some results of Epstein [1] for later reference.

Theorem 1. *Let M^3 be a 3-manifold that possibly has boundary, but that contains no 2-sided projective plane. Let G be a subgroup of $\pi_1(M^3)$ such that G is $*$-indecomposable and not infinite cyclic. Suppose that either*

(a) *G has torsion, or*

(b) *G is isomorphic to the fundamental group of a closed 3-manifold.*

Then there is a compact 3-submanifold $Z^3 \subset \operatorname{Int} M^3$ such that ∂Z^3 is a 2-sphere and G is conjugate in $\pi_1(M^3)$ to a subgroup of finite index in $\pi_1(Z^3)$. In case (a), *$\pi_1(Z^3)$ is finite and Z^3 is orientable.*

Proof. If G has torsion, then it contains a finite subgroup H with more than one element. By [1, Theorem 8.2] there is a compact 3-submanifold $Z^3 \subset \operatorname{Int} M^3$ such that $\partial Z^3 \approx S^2$, $\pi_1(Z^3)$ is finite, and H is conjugate to a subgroup of $\pi_1(Z^3)$. (Z^3 is necessarily orientable.) By the Kurosh subgroup theorem [4, p. 243], G is conjugate to a subgroup of $\pi_1(Z^3)$ or to a subgroup of $\pi_1(M^3 - \text{Int } Z^3)$. Since $H \neq \{1\}$, the former must hold.

Suppose now that $G \neq \{1\}$ is torsion-free but that $G \cong \pi_1(N^3)$, where N^3 is closed. Then N^3 contains no projective plane. Thus, since $\pi_1(N^3)$ is not $\cong Z$ and is $*$-indecomposable, it follows from Epstein's version [1, Theorem 1.1] of the sphere theorem that $\pi_2(N^3) = 0$. Further, $\pi_1(N^3)$ is infinite, so that N^3 is aspherical by [1, Lemma 8.1]. The existence of the required 3-submanifold Z^3 is then a consequence of [1, Theorem 8.8].

There is a variation on Theorem 1 which some may prefer. (Cf. also the definition of "splittable group" given before Theorem 3 and used in Theorems 3 and 4.)

Theorem 1′. *Let M^3 be a 3-manifold that possibly has boundary, but that contains no 2-sided projective plane. Let G be a subgroup of $\pi_1(M^3)$ such that G is $*$-indecomposable, and not infinite cyclic. Suppose that either*

(a) *G has torsion, or*

(b) *$H_3(G; Z_2) \neq 0$ and G embeds in $\pi_1(N^3)$ for some compact 3-manifold N^3.*

Then there is a compact 3-submanifold $Z^3 \subset \text{Int } M^3$ such that ∂Z^3 is a 2-sphere and G is conjugate in $\pi_1(M^3)$ to a subgroup of finite index in $\pi_1(Z^3)$. In case (a), *$\pi_1(Z^3)$ is finite and Z^3 is orientable.*

The proof of Theorem 1′ is similar to that of Theorem 1, and exploits the ideas of §7 of [1]. The proof is left to the reader.

Theorem 2. *Let M^3 be a 3-manifold that possibly has boundary, but that contains no 2-sided projective plane. Let $G \neq \{1\}$ be a finitely-generated subgroup of $\pi_1(M^3)$ such that G is $*$-indecomposable, and G/G' is finite. Then there is a compact 3-submanifold $Z^3 \subset \text{Int } M^3$ such that ∂Z^3 is a 2-sphere and G is conjugate in $\pi_1(M^3)$ to a subgroup of finite index in $\pi_1(Z^3)$. (It follows that $H_1(Z^3)$ is finite, and that Z^3 is orientable.)*

Proof. By Scott [6], G is finitely-presented. We may thus assume without loss of generality that M^3 is compact. It suffices (by Theorem 1) to show that if G is torsion-free, then G is isomorphic to the fundamental group of a closed 3-manifold. By Theorem 4 of [3] (cf. also [2, Theorem 4.4], which contains an alternate proof), $G \cong \pi_1(N^3)$ for some compact 3-manifold N^3. If G is torsion-free, then ∂N^3 contains no projective plane and hence by Lemma 2.4 of [2], ∂N^3 consists entirely of 2-spheres. Then $\hat{N}^3$ is the required closed 3-manifold. Verifying the parenthetical remark is left as an exercise for the reader.

Let us say that a finitely-generated group G is *splittable* if G has no infinite cyclic free factor, and if each free factor of G that is $*$-indecomposable either has torsion, or is isomorphic to the fundamental group of a closed 3-manifold. For example, it suffices to know that G is the fundamental group of a 3-manifold and has finite abelianization. (Cf. also Theorem 1$'$.)

Theorem 3. *Suppose that S is an arcwise-connected Hausdorff space and U is an open, connected set in S such that U is also an open 3-manifold containing no 2-sided projective plane. Let G be a finitely-generated group that is splittable, $G \subset \pi_1(U)$. Suppose that there does not exist a compact 3-submanifold $Z^3 \subset U$ with ∂Z^3 a 2-sphere, and two distinct free factors of G such that each factor is conjugate in $\pi_1(U)$ to a subgroup of finite index in $\pi_1(Z^3)$. Then the inclusion-induced homomorphism $\pi_1(U) \to \pi_1(S)$ maps G monomorphically.*

Proof. Write $G = G_1 * \cdots * G_n$, where each $G_i \neq \{1\}$ is $*$-indecomposable. Then each G_i is finitely-generated, none is isomorphic to Z, and each G_i either has torsion or is isomorphic to the fundamental group of a closed 3-manifold. Then repeated application of Theorem 1 (keeping in mind our special "conjugacy" hypothesis about the G_i's) yields disjoint compact 3-submanifolds $Z_1, \cdots, Z_n$ such that each $\partial Z_i \approx S^2$ and for each i, G_i is conjugate in $\pi_1(U)$ to a subgroup of finite index in $\pi_1(Z_i)$.

Now fix a basepoint $p \in U - \bigcup Z_i$ for $\pi_1(U)$. Then by van Kampen's theorem and the fact that S is Hausdorff (so that $S - \bigcup Z_i$ is open in S),

$$\pi_1(S, p) = \pi_1(Z_1) * \cdots * \pi_1(Z_n) * \pi_1(S - \bigcup Z_i, p),$$

where the embedding of $\pi_1(Z_i)$ in $\pi_1(S, p)$ is determined by translating $\pi_1(Z_i)$ to the basepoint along the ith arc in a disjoint collection of arcs in $U - \bigcup \operatorname{Int} Z_i$. Let k be the inclusion $U \to S$. Then since G_i is conjugate in $\pi_1(U, p)$ to a subgroup of finite index in $\pi_1(Z_i)$, k_* maps each G_i monomorphically into $\pi_1(S, p)$.

By Scott's Theorem 1.7 of [6], the fact that k_* maps G monomorphically will follow once it is shown that the subgroup

$$J = \text{Image}\,[k_*: G \to \pi_1(S, p)]$$

has at least n free factors. If J had fewer than n free factors, some ($*$-indecomposable, $\not\approx Z$) free factor J_1 of J would contain conjugates (in J) of two distinct groups $k_*(G_i)$, say $k_*(G_1)$ and $k_*(G_2)$. But then, recalling the displayed expression for $\pi_1(S, p)$ above, we see that J_1 would be conjugate in

$\pi_1(S, p)$ to a subgroup of exactly one of the free factors displayed there. This is impossible, since a subgroup $\neq \{1\}$ of one free factor cannot also be conjugate in $\pi_1(S, p)$ to a subgroup of another free factor. The result follows.

Example. Let M^3 be the compact, orientable 3-manifold-with-boundary obtained from real projective 3-space P^3 by removing the interior of a 3-cell, then adding an orientable 1-handle to the resulting 2-sphere boundary. Thus,

$$\pi_1(M^3) \cong Z_2 * Z \cong \pi_1(\text{Int } M^3).$$

Let $J \subset \partial M^3$ be a simple closed curve representing (up to conjugacy) the generator t of the Z factor of $\pi_1(M^3)$. Let S be obtained by adding a 2-handle to M^3 along J, so that $\pi_1(S) \cong Z_2$. If $U = \text{Int } M^3$ and G is the subgroup of $\pi_1(U)$ generated by Z_2 and $t^{-1}Z_2t$, then $G \cong Z_2 * Z_2$ (see Problem 10, p. 194 of [4]), yet the inclusion-induced homomorphism $\pi_1(U) \to \pi_1(S)$ does not map G monomorphically. Of course, the "conjugacy" hypothesis of Theorem 3 is violated.

Theorem 4. *Let M^3 be a 3-manifold that possibly has boundary, but that contains no 2-sided projective plane. Suppose that $X \subset M^3$ is a connected compact absolute neighborhood retract whose fundamental group $\pi_1(X)$ is splittable. (For example, suppose that $H_1(X)$ is finite.) Then each sufficiently tight connected neighborhood U of X in M^3 contains a connected neighborhood V of X in M^3 such that each loop in $V - X$ is contractible in U.*

Proof. We may assume that M^3 is an open 3-manifold. Let U be any connected neighborhood of X in M^3 that retracts onto X. Let V be any connected neighborhood of X in U that admits a *strong deformation retraction in U onto X*. (That is, there is a map $h: V \times I \to U$ such that, letting $h_t(x) = h(x, t)$ and identifying $V \times \{0\}$ with V, we have: h_0 = inclusion $V \to U$, each $h_t|X$ = inclusion $X \to U$, and h_1 retracts V onto X.) Write $\pi_1(X) = G_1 * \cdots * G_n$, where each $G_i \neq \{1\}$ is $*$-indecomposable. Then each G_i is finitely-generated, no G_i is isomorphic to Z, and each G_i either has torsion or is isomorphic to the fundamental group of a closed 3-manifold.

We claim that each loop in $V - X$ is contractible in U. Let $L: \partial\Delta^2 \to V - X$ be given. (We put $|L| = L(\partial\Delta^2)$.) Let V_1 be a connected neighborhood of X in $V - |L|$ that admits a strong deformation retraction in $V - |L|$ onto X. Let i, j denote the inclusions $X \to V_1$, $V_1 \to V - |L|$, respectively.

We note first that there is no compact 3-submanifold $Z^3 \subset V_1$ such that $\partial Z^3 \approx S^2$ and such that (say) each of $i_*(G_1)$ and $i_*(G_2)$ is conjugate in

$\pi_1(V_1)$ to a subgroup of finite index in $\pi_1(Z^3)$. (Observe that $\pi_1(Z^3) \not\cong Z$, and that $\pi_1(Z^3)$ is $*$-indecomposable.) For

$$j_*\pi_1(Z^3) \subset j_*\pi_1(V_1) = j_*i_*\pi_1(X) \cong G_1 * \cdots * G_n.$$

Thus, the denial of our assertion would imply that each of $j_*i_*(G_1)$ and $j_*i_*(G_2)$ is conjugate in $\pi_1(V - |L|)$ to a subgroup of the same one of $j_*i_*(G_1), \cdots, j_*i_*(G_n)$ into which $j_*\pi_1(Z^3)$ conjugates. Since $j_*i_*\pi_1(X)$ is a retract of $\pi_1(V - |L|)$, this is impossible.

We can now verify our earlier claim by an application of Theorem 3. Specifically, let S be the space obtained by attaching a 2-cell to U along L, and let the present V_1 be the distinguished open set considered in the statement of Theorem 3. Then L is freely homotopic in U to a loop in X, and hence to a loop $\bar{L}$ in V_1 representing an element of $i_*\pi_1(X)$. By Theorem 3, the inclusion-induced homomorphism $\pi_1(V_1) \to \pi_1(S)$ maps $i_*\pi_1(X)$ monomorphically. Since $\bar{L}$ is contractible in S, it follows that $\bar{L}$ is contractible in V_1. Hence L is contractible in U, as desired.

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