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AN UPPER BOUND FOR THE PERMANENT OF A FULLY INDECOMPOSABLE MATRIX

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ABSTRACT. Let A be an $n \times n$ fully indecomposable matrix with non-negative integer entries and let $\sigma(A)$ denote the sum of the entries of A. We prove that $\operatorname{per}(A) \le 2^{\sigma(A)-2n} + 1$ and give necessary and sufficient conditions for equality to hold.

1. **Introduction.** The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation is over all elements of the symmetric group. If $A = (a_{ij})$ is an $n \times n$ $(0, 1)$ matrix, then $0 \le \operatorname{per}(A) \le \prod_{i=1}^n r_i$ where $r_i = \sum_{j=1}^n a_{ij}$, $i = 1, \cdots, n$. Improvements of this upper bound have been made by several authors; see [1]–[4]. The first was in 1963 by Minc, who showed that $\operatorname{per}(A) \le \prod_{i=1}^n (r_i + 1)/2$, with equality if and only if A is a permutation matrix. Here we give an easily computed upper bound for $\operatorname{per}(A)$ in terms of N, the number of positive entries in A, and n, the dimension of A.

A conjecture of E. J. Roberts [5, p. 78] states that if A is an $n \times n$ nearly decomposable $(0, 1)$ matrix with N positive entries then $\operatorname{per}(A) \le 2^{N-2n} + 1$. We prove a stronger result and determine for which matrices A equality holds.

2. **Results.**

Theorem 1. *Let A be an $n \times n$ $(0, 1)$ matrix with all row sums $r_i \ge 3$. Let N be the number of positive entries in A. Then* $\operatorname{per}(A) < 2^{N-2n}$.

Proof. It can be verified by induction that for $k \ge 3$, $(k + 1)/2 \le 2^{k-2}$. Since $r_i \ge 3$ for all i, A is not a permutation matrix. Hence by Minc's result

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$$\operatorname{per}(A) < \prod_{i=1}^{n} \frac{r_i + 1}{2} \leq \prod_{i=1}^{n} 2^{r_i - 2} = 2^{\Sigma_{i=1}^{n} r_i - 2n} = 2^{N-2n}.$$

For A an $n \times n$ matrix, let $\sigma(A)$ denote the sum of all entries of A. A nonnegative n-square matrix A is said to be *fully indecomposable* if A does not contain an $s \times (n-s)$ zero matrix, $1 \leq s \leq n-1$. As a consequence of the Frobenius-König theorem, every $(n-1)$-square submatrix of such a matrix must have a positive diagonal. Hence, if $A(i|j)$ denotes the submatrix obtained from A by deleting the ith row and jth column, we must have $\operatorname{per} A(i|j) > 0$.

Lemma 1. *Let A be an $n \times n$ fully indecomposable matrix with nonnegative integer entries. Suppose for some i, j $a_{ij} \geq 2$. Then*

$$\operatorname{per}(A) \leq 2 \operatorname{per}(A - E_{ij}) - 1 \tag{1}$$

(E_{ij} denotes the $n \times n$ matrix whose (i, j) entry is 1 and whose other entries are 0).

Proof. Since A is fully indecomposable there is an $l \neq j$ such that $a_{il} \geq 1$. Expanding the permanent by its ith row, we have

$$\begin{aligned}\operatorname{per}(A) &= a_{ij} \operatorname{per}(A(i|j)) + a_{il} \operatorname{per}(A(i|l)) + \sum_{k \neq i,l} a_{ik} \operatorname{per}(A(i|k)) \\ &\geq 2 \operatorname{per}(A(i|j)) + 1\end{aligned}$$

or $\operatorname{per}(A(i|j)) \leq (\operatorname{per}(A) - 1)/2$. Expanding $\operatorname{per}(A)$ by its ith row again, we have

$$\operatorname{per}(A) = \operatorname{per}(A - E_{ij}) + \operatorname{per}(A(i|j)) \leq \operatorname{per}(A - E_{ij}) + (\operatorname{per}(A) - 1)/2$$

from which (1) follows.

The proof shows that equality holds in (1) if and only if $a_{ij} = 2$, $a_{il} = 1$, $\operatorname{per}(A(i|l)) = 1$, and $a_{ik} = 0$ for $k \neq i, l$. Moreover by expanding the permanent of A by the jth column we see that there exists $m \neq i$ such that $a_{mj} = 1$ and $a_{kj} = 0$ for $k \neq m, i$.

Theorem 2. *Let $n \geq 1$ and suppose A is an $n \times n$ fully indecomposable matrix with nonnegative integer entries. Then*

$$\operatorname{per}(A) \leq 2^{\sigma(A) - 2n} + 1. \tag{2}$$

For $n > 1$ equality holds if and only if there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} C_1 & 0 & \cdots & & E_1 \\ E_2 & C_2 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & & & 0 \\ 0 & \cdots & 0 & E_j & C_j \end{bmatrix} \tag{3}$$

where C_i *is* n_i*-square,* E_i *is* $n_i \times n_{i-1}$ $(i = 2, \cdots, j)$ *and* E_1 *is* $n_1 \times n_j$. *Each* E_i *is a* $(0, 1)$ *matrix containing exactly one* 1. *Each* C_i *is the sum of an identity matrix and a full cycle permutation matrix or is* (x_{11}) *where* x_{11} *equals* 1 *or* 2. *For* $n = 1$ *equality holds for the matrix* (a) *where* $a = 2$ *or* $a = 3$.

Proof. We proceed by induction on n. Let A be an $n \times n$ fully indecomposable matrix with nonnegative integer entries. If $n = 1$ the result is easily verified, so we assume $n \geq 2$. If there exists $a_{ij} \geq 2$ set $A_1 = A - E_{ij}$. Then $\sigma(A_1) = \sigma(A) - 1$ and by Lemma 1, $\text{per}(A) \leq 2\ \text{per}(A_1) - 1$. If A_1 has an entry $a_{kl} \geq 2$, set $A_2 = A_1 - E_{kl}$. Then $\text{per}(A_1) \leq 2\ \text{per}(A_2) - 1$ so that $\text{per}(A) \leq 2^2\ \text{per}(A_2) - (2^2 - 1)$.

Repeated application of Lemma 1 gives a sequence $A = A_0, A_1, A_2, \cdots$ of fully indecomposable matrices with nonnegative integer entries such that $\text{per}(A) \leq 2^j\ \text{per}(A_j) - (2^j - 1)$ and $\sigma(A_j) = \sigma(A) - j$ $(j = 0, 1, 2, \cdots)$.

Eventually we obtain a $(0, 1)$ matrix $B = A_m$ $(m \geq 0)$. Let r_i be the ith row sum of B $(i = 1, 2, \cdots, n)$. There are two cases to consider.

(i) $r_i \geq 3$ for all i. Then by Theorem 1 $\text{per}(B) \leq 2^{\sigma(B) - 2n} - 1$ so that

$$\text{per}(A) \leq 2^m(2^{\sigma(B)-2n} - 1) - (2^m - 1) = 2^{\sigma(A)-2n} + 1 - 2^{m+1} < 2^{\sigma(A)-2n} + 1.$$

(ii) There is an i such that $r_i = 2$, say $i = 1$. Let the 1's in row 1 of B be in columns 1 and 2. Form a new matrix B' by adding together columns 1 and 2 of B and deleting row 1. B' is a fully indecomposable $(n - 1)$-square matrix and $\text{per}(B') = \text{per}(B)$, $\sigma(B') = \sigma(B) - 2$. B' has entries in $\{0, 1, 2\}$ so by the induction hypothesis

$$\text{per}(B) = \text{per}(B') \leq 2^{\sigma(B') - 2(n-1)} + 1 = 2^{\sigma(B)-2n} + 1.$$

Hence

$$\text{per}(A) \leq 2^m(2^{\sigma(B)-2n} + 1) - (2^m - 1) = 2^{\sigma(A)-2n} + 1.$$

If equality holds in (2), the proof shows that for $n \geq 2$ the following must hold:

(a) All entries of A are less than or equal to 2 and if $a_{ij} = 2$ then the ith row sum and jth column sum of A are both equal to 3.

(b) B has a row sum equal to 2.

(c) $\operatorname{per}(B') = 2^{\sigma(B')-2(n-1)} + 1$.

By the induction hypothesis there exist permutation matrices P' and Q' such that

$$P'B'Q' = \begin{bmatrix} C_1' & 0 & \cdots & & E_1' \\ E_2' & C_2' & & & 0 \\ & & & & \vdots \\ & \ddots & & \ddots & 0 \\ 0 & \cdots 0 & E_j' & & C_j' \end{bmatrix}$$

where C_i', E_i' are as described in the statement of the theorem. This implies that no row or column sum of B' exceeds 3, so that each row and column of B' has at most three positive entries. Thus B must have only two or three 1's in columns 1 and 2 among the rows $2, 3, \cdots, n$. Hence one of columns 1 and 2 of B has only two 1's. It follows that after permuting rows and columns B must have the form (3). We must now replace certain 1's of B by 2's to obtain A.

Suppose first that some C_i has a 2 (after replacement) and that $n_i > 1$. Without loss of generality we can assume $i = 1$ and that the 2 is in row n_1, column n. Then $a_{1n_1} = a_{n_1,n_1-1} = 1$. Thus

$$A = \begin{bmatrix} C_1 & 0 & \cdots & 0 & E_1 \\ E_2 & C_2 & & & 0 \\ 0 & E_3 & C_3 & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & E_l & C_l \end{bmatrix} \quad \text{where } C_1 = \begin{bmatrix} y_1 & & & & 1 \\ x_2 & y_2 & & & \vdots \\ & \ddots & \ddots & & \\ & & x_{n_1-1} & y_{n_1-1} & 0 \\ & & & 1 & 2 \end{bmatrix}.$$

Then

$$1 = \operatorname{per} A(1|n_1) \geq \prod_{i=2}^{n_1-1} x_i \prod_{j=2}^{l} \operatorname{per}(C_j) \geq 1$$

so that $x_2 = \cdots = x_{n_1-1} = 1$, $C_j = (1)$ $(j = 2, \cdots, l)$ and $l = n - n_1 + 1$. Let e_j be the positive entry of E_j $(j = 1, \cdots, l)$, let e_1 be in row i_0, column n, and let e_2 be in row $n_1 + 1$, column i. We show that $i_0 \leq i$. Suppose $i_0 > i$. Then

$$2^{\sigma(A)-2n} + 1 = \operatorname{per}(A) = 2 \prod_{j=1}^{n_1-1} y_j + 1 + \prod_{j=1}^{l} e_j$$

so that

$$2^{\Sigma y_j + \Sigma e_j + 2 - n} \leq 2^{\Sigma y_j - (n_1-1)+1} + 2^{\Sigma e_j - (n-n_1+1)}$$

or

$$1 \leq 2^{n-n_1-\Sigma e_j} + 2^{n_1-3-\Sigma y_j}.$$

Now $\Sigma\, e_j \geq n - n_1 + 1$ and $\Sigma\, y_j \geq n_1 - 1$ so that $n - n_1 - \Sigma\, e_j \leq -1$ and $n_1 - 3 - \Sigma\, y_j \leq -2$. Hence $1 \leq 2^{-1} + 2^{-2} = 3/4$ which is a contradiction. Thus $i_0 \leq i$. We again calculate

$$2^{\sigma(A)-2n} + 1 = \operatorname{per}(A) = 2 \prod y_i + 1 + 2\left(\prod e_j\right)\left(\prod y_j\right)\frac{1}{Y}$$

where $Y = \Pi_{j=i_0}^{i}\, y_j$. Thus

$$2^{\Sigma e_j + \Sigma y_j + 1 - n} \leq 2^{\Sigma y_j - (n_1-1)} + \frac{1}{Y}\, 2^{\Sigma e_j + \Sigma y_j - n}$$

or

$$1 \leq 2^{n-n_1-\Sigma e_j} + \frac{1}{Y}\, 2^{-1} \leq 2^{-1} + 2^{-1} = 1.$$

Therefore equality must hold throughout. This means that $\Sigma\, e_j = n - n_1 + 1$ so each $e_j = 1$. Also $Y = 1$ so $y_{i_0} = \cdots = y_i = 1$ and $y_j = 1$ or 2 whenever $j < i_0$ or $j > i$. It follows that the submatrix formed by rows $i_0, \cdots, i, n_1+1, \cdots, n$ and colums $i_0, \cdots, i, n_1 + 1, \cdots, n$ is the sum of an identity matrix and a full cycle permutation matrix. Thus we may rearrange the rows and columns of A to get A into the form (3), where each $C_i = (y_i')$, $y_i' = 1$ or 2, $i = 1, \cdots, j-1$, and at least one y_i' will be equal to 2. The matrix C_j will be the sum of an identity matrix and a full cycle permutation matrix.

The remaining case is when each C_i is the sum of an identity matrix and a full cycle permutation matrix or is (x_{11}) where x_{11} equals 1 or 2. Let e_i be the positive entry of E_i $(i = 1, 2, \cdots, j)$. If all e_i's are 1 then A has the form (3). Suppose $e_l > 1$. Then $\sigma(A) > 2n$ and $C_l = C_{l-1} = (1)$. We have

$$2^{\sigma(A)-2n} + 1 = \operatorname{per}(A) = \prod_{k=1}^{j} \operatorname{per}(C_k) + 2 \prod_{k=1;\, k\neq l, l-1}^{j} e_k.$$

Hence $\Pi_{k=1}^{j} \operatorname{per}(C_k)$ is an odd integer. By assumption on the C_i's, it must be that $C_i = (1)$ for all i. But then A can be rearranged to get $A = \operatorname{diag}(e_1, \cdots, e_j) + P$, where $e_i = 1$ or 2 and P is a full cycle permutation matrix. Since A now has the form (3), this completes the proof.

Now suppose A is a (0, 1) matrix with row sums $r_1, r_2, \cdots, r_n$. If all the r_i's are greater than 2 then Minc's result gives a better upper bound for the permanent than Theorem 2. For this case the bound obtained from the result of Nijenhuis and Wilf [4] will be better than Minc's. However, if $r_i = 2$ for some i then Theorem 2 may give a better bound than either of these. For example, the matrix A below has $\operatorname{per}(A) = 3$. Minc's bound is $2(3/2)^3 = 6.25$, the Nijenhuis and Wilf bound is about 7.29, and Theorem 2 gives $\operatorname{per}(A) = 3$.

The matrix $A(\epsilon)$ below (with $\epsilon = 0$) shows that the assumption of full indecomposability in Theorem 2 cannot be dropped. The same example with $\epsilon > 0$ and sufficiently small shows that Theorem 2 does not hold for arbitrary nonnegative matrices.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad A(\epsilon) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & \epsilon \\ 0 & \epsilon & 1 \end{bmatrix}.$$

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