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EXISTENCE THEOREMS FOR URYSOHN'S INTEGRAL EQUATION

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ABSTRACT. The theory of abstract Hammerstein operators is applied to obtain existence theorems for Urysohn's integral equation.

Urysohn's integral equation is of the form

$$(*) \qquad u(s) + \int_\Omega \Phi(s, t, u(t))dt = 0.$$

Usually one assumes that Ω is a subset of R^n, and that $\Phi(s, t, u)$ is a function of three variables $s, t \in \Omega, u \in R$, satisfying the so-called Carathéodory conditions. Urysohn's equation has been discussed by Urysohn [6], Kolomý [4], Krasnosel'skiĭ [5] and others. Attempts have been made to apply the theory of monotone operators to get existence theorems for $(*)$. In this paper we apply the theory of abstract Hammerstein operators to obtain existence theorems for $(*)$ with rather simple conditions on the function Φ.

We define a linear operator A: $L^2(\Omega \times \Omega) \to L^2(\Omega \times \Omega)$ with range in $L^2(\Omega)$ and a nonlinear operator F: $L^2(\Omega) \to L^2(\Omega \times \Omega)$ as follows:

$$(1) \qquad [Au](s) = \int_\Omega u(s, t)dt,$$

$$(2) \qquad [Fu](s, t) = \Phi(s, t, u(t)).$$

In all our considerations in this paper, Ω will be a set of finite measure in R^n and

$$(3) \qquad L^2(\Omega) = \left\{u: \int_\Omega |u(t)|^2 dt < \infty\right\},$$

$$(4) \qquad L^2(\Omega \times \Omega) = \left\{u: \int_\Omega \int_\Omega |u(s, t)|^2 dt\, ds < \infty\right\}.$$

Observe that $L^2(\Omega)$ is a closed subspace of $L^2(\Omega \times \Omega)$.

Lemma 1. *A is a continuous linear map from $L^2(\Omega \times \Omega)$ to $L^2(\Omega \times \Omega)$ with range in $L^2(\Omega)$.*

One of the hypotheses of the existence theorem is the compactness of the operator AF. In the following lemmas conditions are given which assure this.

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Lemma 2. *If Φ satisfies the Carathéodory conditions and*

$$|\Phi(s, t, u)| \leq a(s, t) + b(s, t)|u|, \tag{5}$$

$$a, b \in L^2(\Omega \times \Omega), \quad b(s, t) > 0, \quad s, t \in \Omega, \quad u \in R,$$

then F is a continuous bounded map from $L^2(\Omega)$ to $L^2(\Omega \times \Omega)$.

Now we define an operator U: $L^2(\Omega) \to L^2(\Omega)$ by

$$[Uu](s) = \int_\Omega \Phi(s, t, u(t))dt. \tag{6}$$

Obviously $U = AF$. The operator U is generally called Urysohn's integral operator.

Lemma 3. *Under the conditions of Lemma 2 the Urysohn's operator U is a continuous and compact mapping from $L^2(\Omega)$ to $L^2(\Omega)$.*

We shall make use of the following theorem which is a slight variation of Amann's theorem [1].

Theorem 1 (Amann). *Let X be an arbitrary Banach space and A: $X \to X^*$ be an angle-bounded map with constant $\alpha \geq 0$. Let Y be a closed subspace of X^* which contains the range of A. Let F: $Y \to X$ be continuous, bounded and assume that there exists $\rho_0 > 0$ such that for all $u \in R(A)$*

$$\langle u, Fu \rangle \geq -(1 + \alpha^2)^{-1}\|A\|^{-1}\|u\|^2 \tag{7}$$

where $\|u\| > \rho_0$.

If the composite operator AF is compact, then the Hammerstein equation

$$u + AFu = 0 \tag{**}$$

has a solution u in Y such that $\|u\| \leq \rho_0$.

We are now in a position to state and prove our existence theorem.

Theorem 2. *Assume $\Phi(s, t, u)$ satisfies the Carathéodory condition and that the operators A, F are defined as in* (1), (2) *and the map AF from $L^2(\Omega)$ to $L^2(\Omega)$ is compact. Also assume that $\sup_{|u| \leq \sigma}|\Phi(s, t, u)|$ is in $L^1(\Omega)$, where $\sigma > 0$ is such that*

$$u\Phi(s, t, u) \geq -c(s, t)|u|^2 \quad \text{for } |u| > \sigma, \tag{8}$$

$c \in L^{2/(2-r)}$ for some $r \leq 2$; $c(s, t) \geq 0$ for $s, t \in \Omega$.

If ρ_0 is such that

$$\sigma a(\sigma)\rho_p^{-2} + \|c\||\Omega|^{r/2}\rho_0^{r-2} < 1 \tag{9}$$

then the Urysohn's integral equation

$$(*) \qquad u(s) + \int_\Omega \Phi(s, t, u(t))dt = 0$$

has a solution u in $L^2(\Omega)$ such that $\|u\| \le \rho_0$. Here $a(\sigma)$ denotes the L^1 norm of $\sup_{|u|\le r}|\Phi(s, t, u)|$, $\|c\|$ the $L^{2/(2-r)}$ norm of c (the L^∞ norm of c if $r = 2$), $\|u\|$ the L^2 norm of u.

Proof. The assertion will follow from Theorem 1. We set $X = L^2(\Omega \times \Omega)$ and $Y = L^2(\Omega)$. Then $(*)$ is equivalent to the operator equation

$$(**) \qquad u + AFu = 0$$

where A and F are as defined in (1) and (2) respectively. By Lemma 1, A is a linear bounded operator from X to X^* with range contained in Y. And similarly, by Lemma 2, F is a continuous bounded operator from Y to X^*. Also by assumption AF is compact as a map from Y to Y. Furthermore we have

$$\begin{aligned}\langle Au, u\rangle &= \iint_{\Omega\times\Omega} ds\,dt\,u(s, t) \int_\Omega u(s, \tau)d\tau \\ &= \int_\Omega ds \left(\int_\Omega u(s, t)dt\right)\left(\int_\Omega u(s, \tau)d\tau\right) \\ &= \int_\Omega ds \left(\int_\Omega u(s, t)dt\right)^2 \ge 0\end{aligned}$$

which implies that A is monotone as a map from X to X^*. Also $\langle Au, v\rangle = \langle Av, u\rangle$.

Since A is symmetric and monotone, it follows by [2, p. 1348] that A is angle-bounded with constant $\alpha = 0$. Furthermore, using (8) we have

$$\begin{aligned}\langle v, \tilde{F}v\rangle &= \int_\Omega \int_\Omega v(s, t)\Phi(s, t, v(s, t))dsdt \\ &= \int_\Omega \int_{t:|v(s,t)|>\sigma} v(s, t)\Phi(s, t, v(s, t))dt \\ &\quad + \int_{M=\{t:|u(s,t)|\le\sigma\}} v(s, t)\Phi(s, tv(s, t))dtds \\ &\ge -\int_\Omega \int_\Omega |v(s, t)|^2|c(s, t)|dtds - \sigma \int_\Omega \int_M |\Phi(s, t, v(s, t))|dtds \\ &\ge -\|v\|^r\left(\int_\Omega \int_\Omega |c(s, t)|^{2/(2-r)}\right)^{(2-r)/2} \\ &\quad - \sigma \int_\Omega \int_\Omega \sup_{|v|\le\sigma} |\Phi(s, t, v)|dtds \\ &\ge -\|v\|^r\|c\| - \sigma a_1(\sigma)\end{aligned}$$

where

$$a_1(\sigma) = \int_\Omega \int_\Omega \sup_{|v| \le \sigma} |\Phi(s, t, v)| dt ds$$

and $\|v\|$ denotes the $L^2(\Omega \times \Omega)$ norm of v. Thus we have

$$\langle v, \tilde{F}v \rangle \ge -\|v\|^2 \|c\| - \sigma a_1(\sigma).$$

For $v = u \in L^2(\Omega)$, we have $\|v\| = |\Omega|^{1/2}\|u\|$, $\tilde{F}v = Fu$,

$$a_1(\sigma) = a(\sigma) = \int_\Omega \int_\Omega \sup_{|u| < \sigma} |\Phi(s, t, u)| dt ds$$

and $\langle u, Fu \rangle \ge -\|u\|^r |\Omega|^{r/2} \|c\| - \sigma a(\sigma)$, so

$$\langle u, Fu \rangle / \|u\|^2 \ge -|\Omega|^{-1}[\sigma a(\sigma)\|u\|^{-2} + \|c\| |\Omega|^{r/2} \|u\|^{r-2}].$$

Here, on the l.h.s., $\| \ \|$ refers to the $L^2(\Omega \times \Omega)$ norm, whereas on the right it refers to the $L^2(\Omega)$ norm. Hence using (9) we get

$$\langle u, Fu \rangle / \|u\|^2 > -\|A\|^{-1} \quad \text{for } \|u\| > \rho_0.$$

Since the operators A and F satisfy all the conditions of Theorem 1, it follows that (**) has a solution u in Y with $\|u\| \le \rho_0$. This in turn implies that (*) has a solution u in L^2 satisfying $\|u\| \le \rho_0$.

Remark 1. (9) is satisfied for all sufficiently large ρ_0 if either $r < 2$ or $r = 2$ and $\|c\|_\infty |\Omega| < 1$. In these two cases (*) has a solution in L^2.

Corollary 1. *Assume that $\Phi(s, t, u)$ satisfies the Carathéodory conditions and*

$$|\Phi(s, t, u)| \le a(s, t) + b(s, t)|u| \quad \text{for } u \in R,$$

(10)

$$a, b \in L^\infty, \qquad b(s, t) > 0 \quad \text{for } s, t, \in \Omega,$$

(11) $$\|b\|_\infty |\Omega| < 1.$$

Then (*) has a solution u in L^2.

Proof. (10) gives

$$|u||\Phi(s, t, u)| \le |a(s, t)||u| + b(s, t)|u|^2$$

$$= |u|^2[|a(s, t)|/|u| + b(s, t)] \quad \text{if } |u| > \rho_0.$$

So we get

$$u\Phi(s, t, u) \ge -|u|^2[|a(s, t)|/\rho_0 + b(s, t)] \quad \text{if } |u| > \rho_0.$$

In view of condition (10) the composite operator AF is compact by Lemma 3. The result then follows by Theorem 1, Remark 1, since (11) implies

that $(\rho_0^{-1}\|a\|_\infty + \|b\|_\infty)|\Omega|^2 < 1$ for all sufficiently large ρ_0.

Remark 2. We note that condition (10) alone is not sufficient to guarantee the existence of solutions, as we see in the following example.

Example. $\Phi(s, t, u) = a + bu$. Then in 1-dimensional space $X = R^1$, (**) is given by

$$u + a + bu = 0 \quad \text{or} \quad a + (1 + b)u = 0. \tag{12}$$

Φ satisfies the condition (10) but for the existence of solution of (12) for arbitrary a it is necessary that $b \neq -1$.

Also as a corollary to the above theorem we obtain the following existence theorem for the integral equation

$$u(s) + \int_\Omega K_1(s, t)\Phi_1(t, u(t))dt + \int_\Omega K_2(s, t)\Phi_2(t, u(t))dt = 0 \tag{13}$$

which contains a sum of Hammerstein integral operators.

Corollary 2. *Suppose the kernels $K_1(s, t)$ and $K_2(s, t)$ are in $L^\infty(\Omega \times \Omega)$. Also assume that the functions $\Phi_1(s, t)$, $\Phi_2(s, t)$ satisfy the Carathéodory conditions and*

$$\begin{aligned} &|\Phi_1(t, u)| \leq a_1(t) + b_1(t)|u| \quad \text{for } u \in R, \\ &\qquad\qquad a_1, b_1 \in L^2, \quad b_1(t) > 0 \quad \text{for } t \in \Omega, \\ &|\Phi_2(t, u)| \leq a_2(t) + b_2(t)|u| \quad \text{for } u \in R, \\ &\qquad\qquad a_2, b_2 \in L^2, \quad b_2(t) > 0 \quad \text{for } t \in \Omega. \end{aligned} \tag{14}$$

$$\begin{aligned} u\Phi_1(t, u) &\geq -c_1(t)|u|^2 \quad \text{for } |u| > \sigma_1 > 0, \\ u\Phi_2(t, u) &\geq -c_2(t)|u|^2 \quad \text{for } |u| > \sigma_2 > 0, \end{aligned} \tag{15}$$

$c_1, c_2 \in L^{2/(2-r)}$ for some $r \leq 2$, $c_1(t) \geq 0$, $c_2(t) \geq 0$ for $t \in \Omega$. If ρ_0 is a positive number such that

$$[a\sigma + b\sigma^2]\rho_0^{-2} + c|\Omega|^{r/2}\rho_0^{r-2} < 1 \tag{16}$$

then the integral equation (12) has a solution u in L^2 such that $\|u\| \leq \rho_0$.

Here

$$\sigma = \max(\sigma_1, \sigma_2), \qquad a = [\|K_1\|_\infty\|a_1\| + \|K_2\|_\infty\|a_2\|]|\Omega|;$$

$$b = [\|K_1\|_\infty\|b_1\| + \|K_2\|_\infty\|b_2\|]|\Omega|; \qquad c = \|K_1\|_\infty\|c_1\| + \|K_2\|_\infty\|c_2\|;$$

$\|a_1\|$, $\|a_2\|$, $\|b_1\|$, $\|b_2\|$ *denote the* L^1 *norm,* $\|c_1\|$, $\|c_2\|$ *the* $L^{2/(2-r)}$ *norm,* $\|u\|$ *the* L^2 *norm of the respective functions.*

Proof. Set $\Phi(s, t, u) = K_1(s, t)\Phi_1(t, u) + K_2(s, t)\Phi_2(t, u)$; then

$$|\Phi(s, t, u)| = |K_1(s, t)||\Phi_1(t, u)| + |K_2(s, t)||\Phi_2(t, u)|$$
$$\leq [\|K_1\|_\infty a_1(t) + \|K_2\|_\infty a_2(t)] + [\|K_1\|_\infty b_1(t) + \|K_2\|_\infty b_2(t)]|u|.$$

So

$$\int_\Omega \int_\Omega \sup_{|u|\leq\sigma} |\Phi(s, t, u)|dsdt \leq [\|K_1\|_\infty\|a_1\| + \|K_2\|_\infty\|a_2\|]|\Omega|$$
$$+ [\|K_1\|_\infty\|b_1\| + \|K_2\|_\infty\|b_2\|]\sigma|\Omega|$$
$$= a + b\sigma$$

and

$$u\Phi(s, t, u) \geq -|K_1(s, t)|c_1(t)|u|^r - |K_2(s, t)|c_2(t)|u|^r$$
$$\geq -[\|K_1\|_\infty c_1(t) + \|K_2\|_\infty c_2(t)]|u|^r \quad \text{for } |u| > \sigma.$$

Defining the operators A, F as in (1), (2), it follows from (14) and Lemma 3 that the map AF is compact. Now the result follows from Theorem 2.

Remark 3. Conditions (14) and (15) are rather simpler than those of Browder [3] for (12).

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