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TEST MODULES

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ABSTRACT. The results of this paper arose from an investigation of the class of Σ-modules, i.e. those modules M for which $\mathrm{Hom}_R(M, -)$ commutes with direct sums. A module T is called a test module if $\mathrm{Hom}_R(M, -)$ commutes with direct sums of copies of T only when M is a Σ-module. Test modules are characterized and their relation to cogenerators is investigated.

Throughout N will denote the set of natural numbers, R will denote an associative ring with identity, and module will mean unitary left R-module. For modules L and M and indexing set I, $L^{(I)}$ will denote the direct sum of $|I|$ copies of L and, for convenience, $\mathrm{Hom}_R(M, L)$ will be written $\mathrm{Hom}(M, L)$.

The modules M for which $\mathrm{Hom}(M, -)$ commutes with direct sums have been called Σ-modules by Rentschler [5]. A systematic study of Σ-modules is given in his thesis [4]. Σ-modules have been considered by at least three other authors [1, p. 54], [2], and [3].

It follows from the definition that M is a Σ-module if and only if, for each family of modules $\{L_i \mid i \in I\}$ and for each R-homomorphism $f\colon M \longrightarrow \bigoplus\{L_i \mid i \in I\}$, $\pi_i f = 0$ for all but a finite number of $i \in I$. We will consistently use $\pi_i\colon \bigoplus\{L_i \mid i \in I\} \to L_i$ to denote the obvious projection map. It is possible to place certain restrictions on the families $\{L_i \mid i \in I\}$ which must be considered. It is only necessary to consider families, each of whose members is an injective module; the indexing set I may be taken to be countable. The following theorem gives a further reduction which is useful.

Theorem 1. *A module M is a Σ-module if and only if, for each module L,* $\mathrm{Hom}(M, -)$ *commutes with direct sums of the module L.*

Proof. The "only if" part is trivial. For the "if" part, begin with a family $\{L_i \mid i \in I\}$ of modules; set $L = \bigoplus\{L_i \mid i \in I\}$; and let $\mu_i\colon L^{(I)} \longrightarrow L$ denote the projection map. Now let $f \in \mathrm{Hom}(M, L)$ and define $\bar{f}\colon M \longrightarrow$

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$L^{(I)}$ via $(\pi_i \mu_j \bar{f})(m) = y \in L_i$, where $y = 0$ if $i \neq j$ and $y = (\pi_i f)(m)$ if $i = j$. $\bar{f}$ is a homomorphism and the assumption yields a finite subset J of I such that if $j \in I - J$, $(\mu_j \bar{f})(M) = 0 \in L$. If $(\pi_i f)(M) \neq 0$, then $(\pi_i \mu_i \bar{f})(M) \neq 0$ so $(\mu_i \bar{f})(M) \neq 0$ and it follows that $i \in J$. This shows that M is a Σ-module.

Remark. It can be shown that one need consider only countable direct sums of the various modules L.

This theorem suggests the question: Is there one module T so that if $\mathrm{Hom}(M, -)$ commutes with direct sums of T then M is a Σ-module? Such a module T would serve as a "test module" for Σ-modules. In fact we adopt this as our definition of a *test module*. We will show next that test modules (always) exist and are quite familiar modules.

Theorem 2. *A module T is a test module if and only if, for each module $X \neq 0$, $\mathrm{Hom}(X, T) \neq 0$.*

Proof. Suppose T is a test module and $\mathrm{Hom}(X, T) = 0$ for a module X. Then $\mathrm{Hom}(X^{(N)}, T) = 0$ so $X^{(N)}$ is a Σ-module. This is impossible if $X \neq 0$. Thus $X = 0$.

Conversely, suppose T is a module satisfying: For each module $X \neq 0$, $\mathrm{Hom}(X, T) \neq 0$. Further assume that X is a module such that $\mathrm{Hom}(X, -)$ commutes with direct sums of T. We must show that X is a Σ-module. Consider any module L and $f \in \mathrm{Hom}(X, L^{(N)})$. Assume, by way of contradiction, that the set $K = \{n \mid n \in N \text{ and } (p_n f)(X) \neq 0\}$ is an infinite set, where p_n: $L^{(N)} \longrightarrow L$ is the nth projection. For each $k \in K$, select $0 \neq h_k \in \mathrm{Hom}(p_k f(X), M)$. If $n \in N$ and $n \notin K$ let $h_n = 0$: $p_n f(X) \longrightarrow M$. If $k \in K$ there exists $x_k \in X$ such that $h_k(p_k(f(x_k))) \neq 0$. Now put $h = \bigoplus_{n \in N} h_n$: $\bigoplus_{n \in N} p_n f(X) \longrightarrow M^{(N)}$. One easily checks that $hf \in \mathrm{Hom}(X, M^{(N)})$. Showing $\pi_k(hf) \neq 0$ if $k \in K$ will contradict the fact that $\mathrm{Hom}(X, -)$ commutes with direct sums of T.

Let $k \in K$,

$$hf(x_k) = h(f(x_k)) = h((p_n f(x_k))) = \left(\bigoplus h_n\right)(p_n f(x_k)) = (h_n(p_n(f(x_k)))).$$

From above the kth component is nonzero. Thus the kth projection of hf is nonzero. With the help of Theorem 1, this completes the proof.

Corollary. *A cogenerator (for the category of left R-modules) is a test module.*

This shows, in answer to the question above, that test modules (always) exist but it raises another question. When is a test module a cogenerator? Before giving the answer we require the following fact.

Lemma. *For a module M there is a submodule H of M and a simple* *module S such that M/H can be embedded in $I(S)$, the injective hull of S.*

Proof. Choose $K \subseteq L \subseteq M$ with L/K simple. If $L/K \subseteq M/K$ is not essential, choose $H/K \subseteq M/K$ such that $H/K \cap L/K = 0$ and H/K is maximal with respect to this property. Then $(L + H)/H$ is simple and essential in M/H.

The next theorem may be of independent interest.

Theorem 3. *For a ring R the following are equivalent:*

(a) *every test module is a cogenerator;*

(b) *for each simple module S, and each submodule $L \subseteq I(S)$, $I(S)/L$ contains an isomorphic copy of $I(S)$.*

Proof. Assume (b) holds. Let C be a test module and consider a simple module $S \neq 0$. By Theorem 2 we choose $0 \neq f \in \mathrm{Hom}(I(S), C)$. By hypothesis $I(S) \hookrightarrow I(S)/\mathrm{Ker}\, f \hookrightarrow C$ so C is a cogenerator.

Now assume (b) fails. Then for some simple module S, we have $N \subseteq I(S)$ such that $I(S)/N$ does not contain a copy of $I(S)$. Let

$$C = (I(S)/N) \oplus \left(\bigoplus \{I(U) \mid U \text{ is simple and } U \ncong S\}\right) \oplus \left(\bigoplus \{M \mid M \subsetneqq I(S)\}\right).$$

C does not contain a copy of $I(S)$ so is not a cogenerator. However, we will show that C is a test module by using Theorem 2.

Let $X \neq 0$ be a module. By the Lemma we choose a simple module U such that $X/Y \subseteq I(U)$ for some submodule $Y \subseteq X$. If $U \ncong S$ then, trivially, $\mathrm{Hom}(X, C) \neq 0$. We consider the two cases (1) $X/Y \cong I(S)$, (2) $X/Y \subset I(S)$, but $X/Y \ncong I(S)$. In the first case, use $I(S)/N$ to get the nonzero element of $\mathrm{Hom}(X, C)$; and, in the second case, use one of the M's, $M \subsetneqq I(S)$. This completes the proof.

The authors would like to thank Professor E. Enochs for the clever construction in the proof of Theorem 3. We note that Tiwary [6] and Vamos [7] have shown that, over an integral domain R, $I(S) \cong I(S)/K$ for all simple modules S and all submodules $K \subseteq I(S)$, if and only if, R_p is a PID for all prime ideals P of R. Thus, for example, over a Dedekind domain a test module is a cogenerator.

The condition (b) of Theorem 3 appears to be interesting. Among the things it implies are: The socle of $I(S)/K$, $K \subseteq I(S)$, consists of copies of S and is essential in $I(S)/K$.

REFERENCES

1. H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968. MR 40 #2736.

2. T. Cheatham, *Direct sums of torsion-free covers*, Canad. J. Math. 25 (1973), 1002–1005. MR 48 #2186.

3. T. Head, *Preservation of coproducts by* $\mathrm{Hom}_R(M, -)$, Rocky Mountain J. Math. 2 (1972), 235–237. MR 45 #3458.

4. R. Rentschler, *Die Vertauschbarkeit des Hom-functor mit directen Summen*, Docktoral These, University München, Munich, 1967.

5. ———, *Sur les modules M tels que* $\mathrm{Hom}(M, -)$ *commute avec les sommes directes*, C. R. Acad. Sci. Paris Ser. A-B 268 (1969), A930–A933. MR 39 #2806.

6. A. K. Tiwary, *On the quotients of indecomposable injective modules*, Canad. Math. Bull. 9 (1966), 187–190. MR 34 #4293.

7. P. Vámos, *A note on the quotients of indecomposable injective modules*, Canad. Math. Bull. 12 (1969), 661–665. MR 41 #190.

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