PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, July 1975

NONCONTINUITY OF TOPOLOGICAL ENTROPY OF MAPS OF THE CANTOR SET AND OF THE INTERVAL

LOUIS BLOCK

ABSTRACT. We show that topological entropy, as a map on the space of continuous functions of the Cantor set into itself, is not continuous anywhere. Furthermore, topological entropy, as a map on the space of continuous functions of the interval into itself, is not continuous at any map with finite entropy.

1. **Introduction.** For a compact topological space S, let $C^0(S, S)$ denote the space of continuous functions of S into itself with the topology of uniform convergence. Let I denote the unit interval $[0, 1]$ and C the Cantor set (the usual middle third Cantor set). For $f \in C^0(S, S)$, let $\operatorname{ent}(f)$ denote the topological entropy of f as defined in [1]. (We review the definition in §2.) $\operatorname{ent}(f)$ is a nonnegative real number, or ∞, which describes (quantitatively) the action of f considered as a discrete dynamical system.

Our main results are the following:

Theorem A. *The function* ent: $C^0(C, C) \to R \cup \{\infty\}$ *is not continuous anywhere.*

Theorem B. *The function* ent: $C^0(I, I) \to R \cup \{\infty\}$ *is not continuous at any map* f *with* $\operatorname{ent}(f)$ *finite.*

We note that Theorem B is valid with I replaced by the circle S^1. (See remarks at the end of §4.)

Topological entropy has been studied in [4], [5] and [7] in connection with Smale's program [8] for studying the orbit structure of differentiable maps of manifolds. However the definition and basic properties rely only on continuity (see [1]). Thus it seems natural to determine what is true in the continuous case before proceeding to the differentiable case.

There are examples on higher dimensional manifolds (see [6]) to show that entropy is not continuous in the differentiable case. However, for the

Received by the editors April 5, 1974.
AMS (MOS) subject classifications (1970). Primary 54H20; Secondary 58F99.
Key words and phrases. Topological entropy, nonwandering set.

circle or the interval the problem is open. In this connection we mention the following. Let $C^1(M, M)$ denote the space of continuously differentiable maps of a compact manifold M into itself with the C^1 topology.

Theorem C. *The function* ent: $C^1(M, M) \to R$ *is continuous at the identity map of* M.

This follows from Proposition 12 of [5].

Finally we remark that positive results on continuity of entropy would have obvious consequences in the theory of bifurcations of differentiable maps. See [2] for some results in this direction.

2. **Preliminary definitions and results.** We begin by reviewing the definition of topological entropy as defined in [1]. Let X be a compact topological space. For any two open covers $\mathfrak{A}$ and $\mathfrak{B}$ of X, let $\mathfrak{A} \vee \mathfrak{B}$ denote $\{A \cap B \mid A \in \mathfrak{A}$ and $B \in \mathfrak{B}\}$. Let $N(\mathfrak{A})$ denote the number of sets in a subcover of $\mathfrak{A}$ of minimum cardinality.

Let $f \in C^0(X, X)$. For each integer $n > 0$ let

$$M_n(\mathfrak{A}) = N(\mathfrak{A} \vee f^{-1}(\mathfrak{A}) \vee \cdots \vee f^{-n}(\mathfrak{A})).$$

Here $f^{-1}(\mathfrak{A})$ denotes the open cover $\{f^{-1}(A) \mid A \in \mathfrak{A}\}$, and f^n is defined inductively by $f^1 = f$ and $f^n = f^{n-1} \circ f$ for $n > 1$.

Set

$$\operatorname{ent}(f, \mathfrak{A}) = \lim_{n\to\infty} n^{-1} \log M_n(\mathfrak{A}).$$

It is easy to show that this limit exists and is finite (see [1]). Finally we define the topological entropy of f by $\operatorname{ent}(f) = \sup \operatorname{ent}(f, \mathfrak{A})$ where the supremum is taken over all open covers $\mathfrak{A}$ of X.

Next we define the notion of nonwandering set. Let $f \in C^0(X, X)$. A point $x \in X$ is said to be wandering if there is a neighborhood 0 of x such that $f^n(0) \cap 0 = \emptyset$ for each integer $n > 0$. The set of points which are not wandering is called the nonwandering set and denoted $\Omega(f)$. We remark that $\Omega(f)$ is a closed subset of X and $f(\Omega(f)) \subset \Omega(f)$.

The following proposition is proved by Bowen in [4]. Here X is a compact metric space.

Proposition 1. *Let* $f \in C^0(X, X)$. *Then* $\operatorname{ent}(f) = \operatorname{ent}(f \mid \Omega(f))$.

One of the inequalities necessary for Proposition 1 follows immediately from the following basic fact which is proved in [1].

Proposition 2. *Let* $f \in C^0(X, X)$ *and let* K *be a closed subset of* X

such that $f(K) \subset K$. *Then* $\operatorname{ent}(f) \geq \operatorname{ent}(f|K)$.

It follows immediately from the definition that if K is finite and $f \in C^0(K, K)$ then $\operatorname{ent}(f) = 0$. Hence by Proposition 1 we have

Proposition 3. *Let* $f \in C^0(X, X)$. *If* $\Omega(f)$ *is finite then* $\operatorname{ent}(f) = 0$.

From the definition of $\Omega(f)$ it follows that $\Omega(f) \subset \operatorname{Im}(f)$ (the image of f). Hence we have

Proposition 4. *Let* $f \in C^0(X, X)$. *If* $\operatorname{Im}(f)$ *is finite then* $\operatorname{ent}(f) = 0$.

3. **Proof of Theorem A.** We may think of the Cantor set C as the set of infinite sequences $(x_1, x_2, \cdots)$ such that each x_k is 1 or 2. The topology on C is then given by the metric

$$d((x_1, x_2, \cdots), (y_1, y_2, \cdots)) = \sum_{i=1}^{\infty} (2^{-i}) |x_i - y_i|.$$

(Equivalently we are thinking of C as the infinite product of the set $\{1, 2\}$ with the product topology.)

Let $f \in C^0(C, C)$. We have two cases.

Case 1. $\operatorname{ent}(f) > 0$.

Define a sequence (f_k) of functions in $C^0(C, C)$ as follows. Let

$$f_k(x_1, x_2, \cdots) = (y_1, y_2, \cdots, y_{k-1}, y_k, 1, 1, 1, \cdots)$$

where $f(x_1, x_2, \cdots) = (y_1, y_2, \cdots)$. In other words f_k is the function which assigns to a sequence $(x_1, x_2, \cdots)$ the sequence whose first k terms are the first k terms of $f(x_1, x_2, \cdots)$ and whose terms past the kth term are all 1.

Note that the image of the map f_k is a finite set consisting of at most 2^k points. (For example the image of f_2 consists at most of the points $(1, 1, 1, 1, 1, \cdots)$, $(1, 2, 1, 1, 1, \cdots)$, $(2, 1, 1, 1, 1, \cdots)$, and $(2, 2, 1, 1, 1, \cdots)$.) Hence by Proposition 4, $\operatorname{ent}(f_k) = 0$.

It is easy to see that the sequence (f_k) converges uniformly to f. In fact if $\epsilon > 0$, we can choose an integer N large enough to insure that $(\sum_{k=N+1}^{\infty} 2^{-k}) < \epsilon$. Then for $k \geq N$,

$$d(f_k(x_1, x_2, \cdots), f(x_1, x_2, \cdots)) < \epsilon$$

for any $(x_1, x_2, \cdots) \in C$.

Case 2. $\operatorname{ent}(f) = 0$.

Define a sequence (g_k) of functions in $C^0(C, C)$ as follows. Let $f(x_1, x_2, \cdots) = (y_1, y_2, \cdots)$ and set

$$g_k(x_1, x_2, \cdots) = (y_1, y_2, \cdots, y_{k-1}, y_k, x_{k+2}, x_{k+3}, \cdots).$$

In other words $g_k(x_1, x_2, \cdots)$ is the sequence whose first k terms are the same as the first k terms of $f(x_1, x_2, \cdots)$, and whose nth term for $n > k$ is x_{n+1}.

As in Case 1, it is clear that (g_n) converges uniformly to f. We conclude the proof by showing that for each integer $k > 0$, $\mathrm{ent}(g_k) \geq \log(2)$. Fix $k > 0$.

Let O_1 be the set of sequences $(x_1, x_2, \cdots)$ such that $x_{k+1} = 1$. Let O_2 be the set of sequences $(x_1, x_2, \cdots)$ such that $x_{k+1} = 2$. Then $\mathfrak{A} = \{O_1, O_2\}$ is an open cover of C. We will show that $\mathrm{ent}(g_k, \mathfrak{A}) = \log(2)$.

Let $x = (x_1, x_2, \cdots) \in C$. Then

$$x \in O_1 \cap g_k^{-1}(O_1) \iff x_{k+1} = 1 \quad \text{and} \quad x_{k+2} = 1,$$

$$x \in O_1 \cap g_k^{-1}(O_2) \iff x_{k+1} = 1 \quad \text{and} \quad x_{k+2} = 2,$$

$$x \in O_2 \cap g_k^{-1}(O_1) \iff x_{k+1} = 2 \quad \text{and} \quad x_{k+2} = 1,$$

$$x \in O_2 \cap g_k^{-1}(O_2) \iff x_{k+1} = 2 \quad \text{and} \quad x_{k+2} = 2.$$

Thus the sets $O_1 \cap g_k^{-1}(O_1)$, $O_1 \cap g_k^{-1}(O_2)$, $O_2 \cap g_k^{-1}(O_1)$, and $O_2 \cap g_k^{-1}(O_2)$ are pairwise disjoint nonempty subsets of C. Hence $M_1(g_k, \mathfrak{A}) = 4$. It follows in the same way by induction that $M_n(g_k, \mathfrak{A}) = 2^{n+1}$ for each integer $n > 0$. Hence $\mathrm{ent}(g_k, \mathfrak{A}) = \log(2)$. This implies that $\mathrm{ent}(g_k) \geq \log(2)$, and completes the proof of Theorem A.

We remark that since the diameter of $\mathfrak{A} \vee g_k^{-1}(\mathfrak{A}) \vee \cdots \vee g_k^{-n}(\mathfrak{A})$ approaches zero (as $n \to \infty$), it actually follows that $\mathrm{ent}(g_k) = \log(2)$.

4. **Proof of Theorem B.** Let K denote any closed interval on the real line. We may form the middle third Cantor subset of K, which we denote by C, and we may identify points in C with sequences whose terms are all 1 or 2, as in §3.

Let s denote the map in $C^0(C, C)$ defined by $s(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots)$. s is sometimes called the full 2-shift (see [8] for discussion and further references). We will use the following elementary facts (see [1]).

Proposition 5. $\mathrm{ent}(s) = \log(2)$.

Proposition 6. *If* $f \in C^0(X, X)$ *for any compact space* X, *then* $\mathrm{ent}(f^n) = n \cdot \mathrm{ent}(f)$.

We will use the usual metric d on $C^0(I, I)$ which may be defined by

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in I\}.$$

Theorem B. *The function* ent: $C^0(I, I) \to R \cup \{\infty\}$ *is not continuous at any map* f *with* $\mathrm{ent}(f)$ *finite.*

Proof. Let $f \in C^0(I, I)$ with $\mathrm{ent}(f)$ finite. Let $\mathrm{ent}(f) = \log(K)$. Pick an integer $m > 0$, such that $2^m > 2K$.

Let x_0 be a fixed point of f. We assume for simplicity that $x_0 \neq 1$. (The proof can be easily modified for the case $x_0 = 1$.)

Let $\epsilon > 0$. $\exists\ \delta > 0$ such that if $|x - x_0| \leq \delta$ then $|f(x) - x_0| < \epsilon/2$. We may choose δ so that $\delta < \epsilon/2$ and $x_0 + \delta \leq 1$.

We construct a map $g \in C^0(I, I)$ such that $d(f, g) < \epsilon$. We first construct g on the interval $[x_0, x_0 + \delta/2]$ as follows. Let C denote the middle third Cantor subset of the interval $[x_0, x_0 + \delta/2]$. Define g on C by $g = s^m$, where s denotes the full 2-shift as defined above. Note that $g(x_0) = x_0$ and $g(x_0 + \delta/2) = x_0 + \delta/2$ since x_0 is identified with the sequence $(1, 1, 1, \cdots)$ and $x_0 + \delta/2$ is identified with the sequence $(2, 2, 2, \cdots)$. We extend g to the interval $[x_0, x_0 + \delta/2]$ by defining g linearly on each open interval in $[x_0, x_0 + \delta/2] - C$.

Next we extend g to the interval $[x_0, x_0 + \delta]$ by defining g on the interval $[x_0 + \delta/2, x_0 + \delta]$ as follows. Let $g(x_0 + \delta/2) = x_0 + \delta/2$, $g(x_0 + \delta) = f(x_0 + \delta)$, and define g linearly on $[x_0 + \delta/2, x_0 + \delta]$. Finally we extend g to a map in $C^0(I, I)$ by defining $g(x) = f(x)$ for $x \in I - [x_0, x_0 + \delta]$.

Note that

$$\mathrm{ent}(g) \geq \mathrm{ent}(s^m) = \log(2^m) > \log(2K) = \log(2) + \log(K).$$

We must show that for all $x \in I$, $|f(x) - g(x)| < \epsilon$.

If $x \in I - [x_0, x_0 + \delta]$ then $|f(x) - g(x)| = 0$. If $x \in [x_0, x_0 + \delta]$ then

$$|g(x) - f(x)| < |g(x) - x_0| + |f(x) - x_0| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Here we have used the fact that g is defined linearly on $[x_0 + \delta/2, x_0 + \delta]$, and $|g(x_0 + \delta/2) - x_0| < \epsilon/2$, and $|g(x_0 + \delta) - x_0| < \epsilon/2$.

We have constructed a map $g \in C^0(I, I)$ such that $d(f, g) < \epsilon$, and $\mathrm{ent}(g) > \mathrm{ent}(f) + \log(2)$. Since ϵ was arbitrary this completes the proof that ent is not continuous at f. Q.E.D.

We conclude this section by remarking that Theorem B is valid with I replaced by the circle S^1. We use the fact that a dense set of maps in

$C^0(S^1, S^1)$ have periodic points (see [3]).

Let $f \in C^0(S^1, S^1)$ and $\epsilon > 0$. Let $f_1 \in C^0(S^1, S^1)$ such that f_1 has a periodic point and $d(f, f_1) < \epsilon/2$. By modifying the argument of Theorem B, with a periodic orbit replacing the role of the fixed point, we construct a map g with $\operatorname{ent}(g) > \operatorname{ent}(f) + \log(2)$, and $d(f_1, g) < \epsilon/2$. Hence ent is not continuous at f.

5. **An example.** We close by giving an example of $f \in C^0(I, I)$ such that $\operatorname{ent}(f)$ is infinite.

Let K_n denote the interval $[1/(n+1), 1/n]$ for each integer $n > 0$. Define f on each interval K_n as follows. Let C_n denote the middle third Cantor subset of K_n. Let $f = s^n$ on C_n (again s denotes the full 2-shift defined in §4) and extend f to K_n by defining f linearly on each open interval in $K_n - C_n$. We extend f to a map in $C^0(I, I)$ by setting $f(0) = 0$.

It follows from Propositions 2, 5, and 6 that $\operatorname{ent}(f) = \infty$.

REFERENCES

1. R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. 114 (1965), 309–319. MR 30 #5291.

2. L. Block, *Bifurcations of endomorphisms of S^1*, Thesis, Northwestern University, 1973.

3. L. Block and J. Franke, *Existence of periodic points for maps of S^1*, Invent. Math. 22 (1973), 69–73.

4. R. Bowen, *Topological entropy and axiom* A, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 23–41. MR 41 #7066.

5. ———, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), 401–414. MR 43 #469.

6. M. Misiurewicz, *On non-continuity of topological entropy*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), 319–320. MR 44 #4781.

7. M. Shub, *Dynamical systems, filtrations, and entropy*, Bull. Amer. Math. Soc. 80 (1974), 27–41.

8. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), 747–817. MR 37 #3598; erratum, 39, p. 1593.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611