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HOMOTOPY SMOOTHING CERTAIN PL-MANIFOLDS

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ABSTRACT. It is shown any simply connected almost parallelizable PL-manifold of dimension $4k + 2 \neq 2^j - 2 \geq 6$ has the homotopy type of a smooth manifold if and only if M is stably parallelizable.

Introduction. It has long been known that except in dimensions congruent to 2 (mod 4) simply connected finite dimensional h-spaces have the homotopy type of smooth stably parallelizable manifolds. (In addition it can be demonstrated that any stably parallelizable smooth h-space manifold is actually parallelizable.) This paper arises from an attempt to analyze the dimensions congruent to 2 (mod 4). Although no complete answer to the original problem is obtained, we believe that the results obtained here will be useful in further analyzing the problem.

According to Browder and Hirsch [3] any simply connected polyhedron satisfying Poincaré duality for $\dim 4k + 2 \geq 5$ whose Spivak normal fibration is trivial has the homotopy type of a closed PL-manifold M whose normal bundle restricted to $M - pt$ is trivial (i.e. is almost parallelizable). If X is a finite dimensional h-space, the Spivak normal fibration is trivial [4]. Therefore one is led to the study of PL-manifolds which are almost parallelizable and whose stable normal bundle is fiber homotopically trivial. (The second condition on the normal bundle actually is a consequence of the first.) The main result of this paper is that if, in addition, M is simply connected of dimension $4k + 2 \neq 2^j - 2$, $k \geq 1$, then M has the homotopy type of a smooth manifold if and only if the stable normal bundle of M is trivial. That is, we obtain an interpretation of the Kervaire invariant as a PL-bundle, which hopefully will lend itself to geometric analysis, at least in the case of h-spaces.

I. Let M^n be a PL-manifold. From surgery theory one obtains the exact sequence

$$\mathcal{S}(M) \xrightarrow{\eta} [M, G/PL] \xrightarrow{c} L_n(\pi_1(M)),$$

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where $\mathcal{S}(M)$ is the set of homotopy triangulations of M, and $\eta(N, \phi) = (\nu_M - g^*\nu_N, h)$ where g is a homotopy inverse to ϕ and h is a stable trivialization of $\nu_M - g^*\nu_N$ determined by ϕ. For details see [11].

Lemma 1. *If M^n is a PL-manifold, then M has the homotopy type of a smooth manifold if and only if there exists a vector bundle ξ over M and a lift of $\nu_M - \xi$ to a G/PL bundle α with $c(\alpha) = 0$.*

Proof. Since $c(\alpha) = 0$, there exists a PL-manifold N and a homotopy equivalence $\phi: N \to M$ with $\nu_N \sim \phi^*(\nu_M - (\nu_M - \xi)) = \phi^*\xi$. Hence ν_N is a vector bundle and by the smoothing result of Milnor [7], [8], N has a smooth structure compatible with its PL-structure. Conversely if $\phi: N \to M$ is a homotopy equivalence with N smooth, then $\eta(N, \phi) \in [M, G/PL]$ and $\xi = g^*\nu_N$ satisfies the conditions where g is a homotopy inverse to ϕ. □

Theorem 1. *Suppose M^n is a simply connected PL-manifold $n = 4k + 2 \neq 2^j - 2$, $k \geq 1$. Further suppose ν_M is stably fiber homotopically trivial and $\tilde{\nu}_M$ is any lift of ν_M to a G/PL-bundle. Then M^n has the homotopy type of a smooth manifold if and only if $c(\tilde{\nu}_M) = 0$.*

Proof. By Lemma 1, M has the homotopy type of a smooth manifold if and only if there exists a vector bundle ξ over M and a lift of $\nu_M - \xi$ to a G/PL-bundle α with $c(\alpha) = 0$. Consider the G/PL-bundle $\tilde{\nu}_M - \alpha$. This projects to a vector bundle and so is a G/O-bundle. Sullivan and Rourke have shown [10] there exist classes $\kappa_{4j+2} \in H^{4j+2}(G/PL; \mathbf{Z}_2)$ so that if $g: M \to G/PL$ is any G/PL-bundle, $c(g) = \langle V^2(M)g^*(\kappa), [M] \rangle$ where $\kappa = \Sigma\kappa_{4j+2}$ and $V(M)$ is the total Wu-class of M. Since ν_M is fiber homotopically trivial, $V(M) = 1$ and therefore $c(\tilde{\nu}_M - \alpha) = \langle (\tilde{\nu}_M - \alpha)^*\kappa_{4k+2}, [M] \rangle$. Now Madsen, Milgram and Brumfiel [6] have demonstrated that if $4k + 2 \neq 2^j - 2$, κ_{4k+2} maps to zero in $H^*(G/O, \mathbf{Z}_2)$. Therefore $c(\tilde{\nu}_M - \alpha) = 0$. But in this dimension c is a homomorphism [11] and so $0 = c(\tilde{\nu}_M - \alpha) = c(\tilde{\nu}_M) - c(\alpha) = c(\tilde{\nu}_M)$. For the converse choose ξ = trivial bundle. □

Proposition. *Suppose M^n is an almost parallelizable PL-manifold, then ν_M is stably fiber homotopically trivial.*

Proof. Consider the cofibration $M - pt \hookrightarrow M \xrightarrow{d} S^n$. $\nu_M^q | M - pt$ is trivial, so there exists a PL-bundle η^q over S^n with $d^*\eta = \nu_M$. Since ν_M is the Spivak normal fibration of M, there exists a degree one map $u: S^{n+q} \to T\nu_M$. Further d is a degree one map, and so $Td \circ u: S^{n+q} \to T\eta^q$ is a degree one map. But this property characterizes the Spivak normal fibration, so η^q is fiber homotopically equivalent to the stable normal bundle of S^n which is trivial. Therefore $\nu_M = d^*\eta^q$ is fiber homotopically trivial. □

We are now in a position to prove the main result.

Theorem 2. *Let M^n be a closed, simply connected, almost parallelizable PL-manifold with $n = 4k + 2 \neq 2^j - 2$, $k \geq 1$. Then M^n has the homotopy type of a smooth manifold if and only if M^n is stably parallelizable.*

Proof. Since M^n is almost parallelizable, ν_M is stably fiber homotopically trivial. From the Atiyah-Milnor-Spanier theorem [1], [9] it is easy to show there exists a map $\phi: \Sigma^k S^n \to \Sigma^k M_+$ ($M_+ = M$ with a disjoint base point) such that $\phi_*(\Sigma^k[S^n]) = \Sigma^k[M^n]$. Consider the cofibration $M^n - pt \hookrightarrow M^n \xrightarrow{d} S^n$. Since d is a degree one map, it follows from the Hopf theorem that $\Sigma^k d \circ \phi \simeq Id$. Therefore if Y is any k-fold loop space $d^*: [S^n, Y] \to [M^n, Y]$ is a monomorphism onto a direct summand. Since $\nu_M | M - pt = 0$ there exists $\eta \in [S^n, BPL]$ with $d^*\eta = \nu_M$. Since BPL is an infinite loop space [2], $\nu_M = 0$ if and only if $\eta = 0$. As in the proof of the previous proposition, η is stably fiber homotopically trivial and so there exists $\widetilde{\eta} \in [S^n, G/PL]$ lifting η. Browder [5] has shown that in the dimensions $4k + 2 \neq 2^j - 2$; the map $[S^{4k+2}, G/PL] \to [S^{4k+2}, BPL]$ is a monomorphism and therefore $\eta = 0$ if and only if $\widetilde{\eta} = 0$. Now $\pi_{4k+2}(G/PL) \simeq \mathbf{Z}_2$ generated by a map with Arf invariant one, so $\widetilde{\eta} = 0$ if and only if $c(\widetilde{\eta}) = 0$. $\widetilde{\eta} d$ is a lift of ν_M and since d has degree one the formula of Rourke and Sullivan for computing c shows $c(\widetilde{\eta}) = 0$ if and only if $c(\widetilde{\eta} d) = 0$. The result now follows from Theorem 1. □

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THE INTERSECTION MULTIPLICITY OF n-DIMENSIONAL PARACOMPACT SPACES

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ABSTRACT. It is shown that there is an integer $\nu(n) \leq 3^{2n+1} - 1$ such that any n-dimensional paracompact space X has intersection multiplicity at most $\nu(n)$. That is, if $\mathfrak{U}$ is an open cover of X, then there is an open cover $\mathfrak{V}$ refining $\mathfrak{U}$ such that any element of $\mathfrak{V}$ intersects at most $\nu(n)$ elements of $\mathfrak{V}$.

For any open cover $\mathfrak{U}$ of a topological space X define $m(\mathfrak{U})$ to be the maximum number of elements of $\mathfrak{U}$ that any member of $\mathfrak{U}$ can intersect. X is said to have intersection multiplicity at most m if every open cover $\mathfrak{U}$ of X has an open refinement $\mathfrak{V}$ covering X such that $m(\mathfrak{V}) \leq m$. The intersection multiplicity of X is then the least integer m such that X has intersection multiplicity at most m and is denoted $m(X)$. Intersection multiplicity is clearly preserved by homeomorphisms. Also if A is a closed subset of X, then $m(A) \leq m(X)$.

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