

## COMPACT COMPLETELY 0-SIMPLE SEMITOPOLOGICAL SEMIGROUPS

JOHN F. BERGLUND

ABSTRACT. A compact semitopological completely 0-simple semigroup with no proper divisors of zero is a semitopological regular 0-paragroup. A compact semitopological regular 0-paragroup has no proper divisors of zero if either indexing set is connected; however, a compact connected regular 0-paragroup may have zero divisors.

The topological version of the Rees theorem for semigroups without zero was completely established for compact topological<sup>1</sup> semigroups (jointly continuous multiplication) by Wallace [8]. For compact semigroups in which the multiplication is only separately continuous ("semitopological semigroups"), the Rees theorem was first established by Pym [7].

In fact much more is obtained. In the case of a compact topological semigroup  $S$ , there is a minimal ideal  $M(S)$  in  $S$  which is either a single point  $\{0\}$  or a completely simple semigroup which is topologically isomorphic to a compact topological paragroup (Rees matrix semigroup over a group). The case for a compact semitopological semigroup is complicated by the fact that, although a minimal ideal must exist, it need not be closed. (It is customary in this subject matter to use words in their topological sense rather than their algebraic sense.) There is, nonetheless, a compact semitopological paragroup which is the continuous monomorphic image of  $M(S)$ . Moreover, the semigroup  $M(S)^{-}/M(S)$  is a null semigroup. (For details, see [1].) If  $M(S)$  is right simple or left simple, then it is closed.

If  $S$  is only locally compact, one cannot conclude the existence of a minimal ideal in  $S$ . Assuming that  $S$  does contain a completely simple minimal ideal  $M(S)$ , one naturally seeks to establish locally compact analogues to the above statements about minimal ideals in compact semigroups. An early attack on this problem, together with the attendant problem of finding reasonable sufficient conditions for the existence of a minimal ideal, was provided by Mostert [4]. Berglund and Hofmann [1] state that a locally

---

Presented to the Society, April 12, 1974; received by the editors April 1, 1974.  
AMS (MOS) subject classifications (1970). Primary 22A15.

<sup>1</sup> All topological spaces dealt with must be Hausdorff.

compact semitopological completely simple semigroup is topologically isomorphic to a locally compact semitopological parargroup, but their proof is incomplete. At this time the problem in that generality remains unsolved. Owen [5] has, however, provided a complete affirmative solution to the problem for locally compact topological semigroups.

Owen was, in fact, dealing with the problem of whether a locally compact topological completely 0-simple semigroup is topologically isomorphic to a locally compact topological regular 0-parargroup. (Paalman-de Miranda [6] established the Rees theorem for compact topological completely 0-simple semigroups.) Owen proved the following theorem.

**Theorem [Owen].** *Let  $S$  be a locally compact topological completely 0-simple semigroup with no proper divisors of zero. If either the maximal group  $eSe - \{0\}$  or the set of nonzero idempotents  $E(S) - \{0\}$  is connected (in particular, if  $S$  is connected), and if  $0$  is a limit point of  $S$ , then*

- (i)  $E(S) - \{0\}$  is compact, and
- (ii)  $S$  is topologically isomorphic to a locally compact topological regular 0-parargroup.

We seek an analogue of Owen's theorem for compact semitopological completely 0-simple semigroups. The hypothesis that there be no proper divisors of zero will be maintained, but none of the other specialized hypotheses. With similar hypotheses, however, on a compact semitopological regular 0-parargroup, a converse theorem is obtained.

*For the remainder of this discussion, let  $S$  be a compact semitopological completely 0-simple semigroup.*

**Lemma.** *Let  $L_0$  be a 0-minimal left ideal of  $S$ , and let  $L = L_0 - \{0\}$ . Then*

- (i)  $L$  is left simple and has an idempotent generator  $e$ ;
- (ii) the set  $E(L)$  of idempotents in  $L$  is compact;
- (iii) the  $\mathcal{H}$ -class  $G$  of  $e$  in  $S$  is a locally compact topological group;
- (iv)  $L$  is topologically isomorphic to the locally compact left-group  $E(L) \times G$ ; and
- (v) the multiplication in  $L$  is jointly continuous.

**Proof.** Statement (i) is well known [2, pp. 68, 77]. Since  $E(L)$  is a left-zero subsemigroup of  $S$ , it is closed in  $S$ , and thus it is compact. The  $\mathcal{H}$ -class  $G$  of  $e$  is  $G = eSe - \{0\}$ . Since  $eSe$  is compact,  $G$  is locally compact. As the  $\mathcal{H}$ -class of an idempotent,  $G$  is also a group. But a locally

compact semitopological group is a topological group. Moreover, by the fundamental theorem of Ellis (see [1]), the action of  $G$  on  $S$  is jointly continuous. In particular, the function  $\eta: E(L) \times G \rightarrow L$ , given by  $\eta(x, g) = xg$ , is continuous. The inverse function  $\eta^{-1}: L \rightarrow E(L) \times G$  is defined by  $\eta^{-1}(s) = (s(es)^{-1}, es)$ . It is continuous because it is composed of the continuous functions  $t \rightarrow et: L \rightarrow G$ ,  $g \rightarrow g^{-1}: G \rightarrow G$ , and  $(t, g) \rightarrow tg: L \times G \rightarrow L$ . Therefore,  $E(L) \times G$  and  $L$  are homeomorphic. That  $E(L) \times G$  is a topological semigroup is clear.

*Notation.* Let  $X$  and  $Y$  be sets,  $G^0$  a group-with-zero, and  $\sigma: Y \times X \rightarrow G^0$  a (sandwich) function. Denote  $\sigma(y, x)$  by  $[y, x]$ . Define a multiplication on  $X \times G^0 \times Y$  by

$$(1) \quad (x_1, g_1, y_1)(x_2, g_2, y_2) = (x_1, g_1[y_1, x_2]g_2, y_2).$$

With this multiplication  $X \times G^0 \times Y$  is a semigroup. Furthermore,  $X \times \{0\} \times Y$  is an ideal in  $X \times G^0 \times Y$ . Let  $(X, G^0, Y; \sigma)$  denote the Rees quotient semigroup  $(X \times G^0 \times Y)/(X \times \{0\} \times Y)$ .

Recall that a 0-paragroup  $(X, G^0, Y; \sigma)$  is regular if and only if, for each  $x_0 \in X$ , there is an element  $y \in Y$  with  $\sigma(y, x_0) \neq 0$  and for each  $y_0 \in Y$ , there is an element  $x \in X$  with  $\sigma(y_0, x) \neq 0$ . Moreover, a 0-paragroup is completely 0-simple if, and only if, it is regular [2, p. 90].

*Example.* In the absence of a zero, if  $X$  and  $Y$  are locally compact Hausdorff topological spaces,  $G$  a locally compact topological group, and  $\sigma$  a separately continuous function, then  $X \times G \times Y$  with the multiplication (1) is a semitopological semigroup. The situation is not so easy in the presence of a zero, however. To begin with, the topological space  $(X \times G^0 \times Y)/(X \times \{0\} \times Y)$  may not be Hausdorff unless the ideal  $X \times \{0\} \times Y$  is compact. Even then, the 0-paragroup may not be semitopological.

Let  $X = Y = [0, 1]$ . Let  $G^0$  be the group-with-zero of nonnegative real numbers under multiplication. Give  $G^0$  the topology of a circle. Let  $\sigma: Y \times X \rightarrow G^0$  be defined by  $\sigma(y, x) = [y, x] = y + x$ . Then  $\sigma$  is continuous, and  $(X, G^0, Y; \sigma)$  is a regular 0-paragroup and a compact Hausdorff space. The multiplication, however, is not separately continuous. For if  $y_n = 1/n$ ,  $x_n = 0$ , and  $g_n = n$ , then  $(x_n, g_n, y_n)$  converges to the zero of  $(X, G^0, Y; \sigma)$ , but

$$(x_n, g_n, y_n)(0, 1, 0) = (0, n, 1/n)(0, 1, 0) = (0, 1, 0).$$

Theorem 2 will clarify this situation.

**Theorem 1.** *If  $S$  has no proper divisors of zero, then  $S$  is topologically isomorphic to a compact semitopological regular 0-paragroup.*

**Proof.** Let  $e$  be a nonzero idempotent in  $S$ , and let  $G$  be the  $\mathcal{H}$ -class of  $e$ . By the Lemma,  $G$  is a locally compact topological group. Moreover,  $G^0 = G \cup \{0\} = eSe$  is a group-with-zero and a compact subgroup of  $S$ . Let  $X = E(Se) - \{0\}$  and  $Y = E(eS) - \{0\}$ . Let  $\sigma: Y \times X \rightarrow G^0$  be defined by  $\sigma(y, x) = [y, x] = yx$ . Since  $S - \{0\}$  is a subsemigroup of  $S$ , the range of  $\sigma$  is contained in  $G$ . And since  $G$  is a locally compact group, by Ellis's theorem, both of the actions  $(s, g) \rightarrow sg: G^0 \times G \rightarrow G^0$  and  $(g, s) \rightarrow gs: G \times G^0 \rightarrow G^0$  are continuous. Whence,  $X \times G^0 \times Y$  is a semitopological semigroup under the multiplication (1). The ideal  $X \times \{0\} \times Y$  is compact; the space of  $(X, G^0, Y; \sigma)$  is, therefore, compact and Hausdorff. Thus,  $(X, G^0, Y; \sigma)$  is a compact semitopological semigroup.

By the Lemma,  $L$  is a locally compact topological semigroup topologically isomorphic to  $E(L) \times G$  under the mapping  $s \rightarrow (s(es)^{-1}, es)$ . Considering the dual statement about  $R$ , and collapsing the middle term, we get that the function  $\zeta: S \rightarrow (X, G^0, Y; \sigma)$ , defined by

$$\zeta(s) = \begin{cases} (s(ese)^{-1}, ese, (ese)^{-1}s), & s \neq 0, \\ 0, & s = 0 \end{cases}$$

is continuous, if it is continuous at zero. So suppose  $s_\alpha, \alpha \in \Lambda$ , is a net in  $S - \{0\}$  which converges to 0 in  $S$ . Since the maps  $s \rightarrow s(ese)^{-1}: (S - \{0\}) \rightarrow X$  and  $s \rightarrow (ese)^{-1}s: (S - \{0\}) \rightarrow Y$  are continuous, and since  $X$  and  $Y$  are compact, there are subnets of  $x_\alpha = s_\alpha(ese)^{-1}$  and  $y_\alpha = (ese)^{-1}s_\alpha$  which converge. We may assume, therefore, that there are elements  $x \in X$  and  $y \in Y$  with  $x = \lim x_\alpha, y = \lim y_\alpha$  and  $(x, 0, y) = \lim(x_\alpha, es_\alpha e, y_\alpha)$  in  $X \times G^0 \times Y$ . Thus,  $0 = \lim \zeta(s_\alpha)$  in  $(X, G^0, Y; \sigma)$ ; and we have that  $\zeta$  is continuous everywhere. Since  $\zeta$  is bijective [2, p. 94], and  $S$  and  $(X, G^0, Y; \sigma)$  are compact Hausdorff spaces,  $\zeta$  is a topological isomorphism as desired.

**Theorem 2.** *Let  $S = (X, G^0, Y; \sigma)$  be a compact semitopological regular 0-paragroup. If either  $X$  or  $Y$  is connected, then  $S$  has no proper divisors of zero.*

**Proof.** Suppose, for definiteness, that  $X$  is connected. We want to show that

$$\sigma(Y \times X) \subseteq G = G^0 - \{0\}.$$

By way of contradiction, suppose there is an element  $(y_1, x_1) \in Y \times X$  with  $\sigma(y_1, x_1) = 0$ . Since  $(X, G^0, Y; \sigma)$  is regular, there is an element  $x_0 \in X$  with  $\sigma(y_1, x_0) \neq 0$ . Since  $\sigma$  must be separately continuous, we have that the set  $B = \{x \in X: \sigma(y_1, x) = 0\}$  is closed and not all of  $X$ . Since the complement of the closed set  $B$  cannot be closed in the connected space  $X$ , there must be a net  $x_\alpha, \alpha \in \mathbf{A}$ , in  $X - B$  which converges to  $w \in B$ , say. Fix  $y_0 \in Y$ . Then, in  $X \times G^0 \times Y$ ,

$$(w, 0, y_1) = \lim(x_\alpha, [y_1, x_\alpha]^{-1}, y_0)$$

since inversion is a homeomorphism on the topological group  $G$ . But, for any  $\alpha \in \mathbf{A}$ ,

$$(x_1, 1, y_1)(x_\alpha, [y_1, x_\alpha]^{-1}, y_0) = (x_1, 1, y_0),$$

contradicting the assumption that  $(X, G^0, Y; \sigma)$  was semitopological.

**Example.** Let  $T = \{a, b, e, f, 0\}$  have multiplication defined by

$$ab = e = e^2, \quad ba = f = f^2, \quad af = a = ea, \quad fb = b = be,$$

and all other products equal to 0. In other words,  $T$  is the matrix semigroup

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

As before, let  $G^0$  be the nonnegative real numbers under multiplication with the topology of a circle. Then  $T \times G^0$  is a compact semitopological semigroup. Let  $I$  be the compact ideal  $I = (T \times \{0\}) \cup (\{0\} \times G^0)$ . Then the Rees quotient semigroup  $S = (T \times G^0)/I$  is a compact connected semitopological completely 0-simple semigroup with proper divisors of zero.

*Question.* Suppose that  $S$  is a compact connected semitopological completely 0-simple semigroup. Can  $S$  have proper divisors of zero only if 0 is a cutpoint?

**Example.** Let  $X = Y = [0, 1]$ , and let  $G^0$  be the nonnegative real numbers under multiplication with the usual noncompact topology. Let  $\sigma: Y \times X \rightarrow G^0$  be defined by  $\sigma(y, 1) = y$  for all  $y \in Y$ , and by

$$\sigma(y, x) = |y - (1 + \tan^2(\pi x/2))^{-1}|$$

if  $x \neq 1$ . Then  $S = (X, G^0, Y; \sigma)$  is a locally compact connected topological regular 0-paragroup in which 0 is a limit point and there are proper divisors of zero.

With everything as above, but with the topology of the circle on  $G^0$ , one obtains a regular 0-paragroup on a compact Hausdorff space, but the semigroup cannot be semitopological since  $X$  and  $Y$  are connected. The multiplication is one-sided continuous, however. Compact semigroups with multiplication continuous on one side arise naturally among some problems in functional analysis. They have not been studied intensively yet.

## REFERENCES

1. J. F. Berglund and K. H. Hofmann, *Compact semitopological semigroups and weakly almost periodic functions*, Lecture Notes in Math., no. 42, Springer-Verlag, Berlin and New York, 1967. MR 36 #6531.
2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, I, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R. I., 1961. MR 24 #A2627.
3. K. H. Hofmann and P. S. Mostert, *Elements of compact semigroups*, Merrill, Columbus, Ohio, 1966. MR 35 #285.
4. P. S. Mostert, *Comments on the preceding paper of Michael's*, J. Austral. Math. Soc. 4 (1964), 287–288. MR 32 #5779.
5. W. S. Owen, *The Rees theorem for locally compact semigroups*, Semigroup Forum 6 (1973), 133–152.
6. A. B. Paalman-de Miranda, *Topological semigroups*, Math. Centre Tracts, 11, Math. Centrum, Amsterdam, 1964, 169 pp. MR 31 #1663.
7. J. S. Pym, *On the Rees-Suschkewitsch structure theorem*, Proc. Cambridge Philos. Soc. 59 (1963), 563–566. MR 27 #729.
8. A. D. Wallace, *The Rees-Suschkewitsch structure theorem for compact simple semigroups*, Proc. Nat. Acad. Sci. U. S. A. 42 (1956), 430–432. MR 18, 14.

DEPARTMENT OF MATHEMATICAL SCIENCES, VIRGINIA COMMONWEALTH UNIVERSITY, RICHMOND, VIRGINIA 23284