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SUFFICIENT CONDITIONS FOR NONOSCILLATION OF A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

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ABSTRACT. Using a generalization of a variational theorem due to Leighton, the authors obtain sufficient conditions for the second order nonlinear differential equation $(a(t)x')' + q(t)f(x) = r(t)$ to be nonoscillatory. Examples showing the necessity of certain hypotheses are also given.

1. **Introduction.** In [1] Komkov generalized a well-known variational theorem of Leighton [2]. In this paper we apply Komkov's result to obtain sufficient conditions for all solutions of the second order nonlinear equation $(a(t)x')' + q(t)f(x) = r(t)$ to be nonoscillatory. Few results of this type are known for nonlinear equations and the authors know of no such results in case $r(t) \not\equiv 0$.

Our main result is contained in Theorem 2. In Theorem 3 we generalize the form of the above equation. We also include examples showing that some of our hypotheses are necessary.

2. **Sufficient conditions for nonoscillation.** Consider the equations

$$(a(t)x')' + q(t)x = 0 \tag{1}$$

and

$$(a(t)x')' + q(t)f(x) = r(t) \tag{2}$$

where a, q, r: $[t_0, \infty) \longrightarrow R$ and f: $R \longrightarrow R$ are continuous and $a(t) > 0$. For reference, we now state Komkov's result [1, Theorem 2].

Theorem. *Suppose there exists a C^1 function $u(t)$ defined on $[t_1, t_2]$ and a function $G(u)$ such that $G(u(t))$ is not constant on $[t_1, t_2]$, $G(u(t_1)) = G(u(t_2)) = 0$, $g(u) = G'(u)$ is continuous,*

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$$\int_{t_1}^{t_2} \{a(t)[u'(t)]^2 - q(t)G(u(t))\}\,dt < 0,$$

and $g^2(u(t)) \leq 4G(u(t))$ *for* t *in* $[t_1, t_2]$. *Then every solution of* (1) *must vanish on* $[t_1, t_2]$.

We classify solutions of (2) in the following way. A solution $x(t)$ of (2) will be called nonoscillatory if there exists $t_1 \geq t_0$ such that $x(t) \neq 0$ for $t \geq t_1$; the solution will be called oscillatory if for any given $t_1 \geq t_0$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$, $x(t_2) > 0$ and $x(t_3) < 0$; and it will be called a Z-type solution if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

Theorem 1. *Suppose that*

(i) *equation* (1) *is nonoscillatory,*

(ii) $r(t) \geq 0$ *and* $r(t) \not\equiv 0$ *on any interval,*

(iii) $xf(x) \geq 0$ *for all* x *and* $f(x) \not\equiv 0$ *on any interval, and*

(iv) $[d(xf(x))/dx]^2 \leq 4xf(x)$ *for all* $x \geq 0$.

Then no solution of (2) *is oscillatory or nonnegative* Z*-type.*

Proof. Suppose that $u(t)$ is an oscillatory or nonnegative Z-type solution of (2) and let t_1 and t_2 be consecutive zeros of $u(t)$ such that $t_1 < t_2$ and $u(t) > 0$ for t in (t_1, t_2). Since $u(t)$ is a solution of (2), we have

$$(a(t)u'(t))'u(t) + q(t)f(u(t))u(t) = r(t)u(t)$$

or,

$$(a(t)u'(t)u(t))' - r(t)u(t) = a(t)[u'(t)]^2 - q(t)f(u(t))u(t).$$

Hence

$$\int_{t_1}^{t_2} \{a(t)[u'(t)]^2 - q(t)f(u(t))u(t)\}\,dt < 0.$$

Letting $G(u) = f(u)u$, we see that the hypotheses of Komkov's theorem are satisfied so equation (1) is oscillatory. This contradiction completes the proof.

Example 1. The equation

$$x'' - 9f(x) = 3(\sin t - 1)(-3\sin^2 t + \sin t - 1), \qquad t \geq 0,$$

where

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ x^{1/3}, & \text{if } x < 0, \end{cases}$$

satisfies the hypotheses of our theorem but does possess the nonpositive Z-type solution $x(t) = (\sin t - 1)^3$.

By putting a condition on the sign of $q(t)$ we can eliminate any nonpositive Z-type solutions and thus obtain the following nonoscillation result.

Theorem 2. *If, in addition to the hypotheses of Theorem* 1, $q(t) \geq 0$ *for* $t \geq t_0$, *then all solutions of* (2) *are nonoscillatory.*

Proof. It will suffice to show that (2) cannot have nonpositive Z-type solutions. Suppose that $x(t)$ is such a solution. Then

$$(a(t)x'(t))' = r(t) - q(t)f(x(t)) \geq 0.$$

Letting t_1 be a zero of $x'(t)$ and integrating, we have $a(t)x'(t) \geq 0$ for $t \geq t_1$, so $x'(t) \geq 0$ for $t \geq t_1$ which is impossible for Z-type solutions.

The next example shows that $r(t)$ cannot be allowed to change signs.

Example 2. The equation

$$x'' + x/t^3 = [t \cos(\ln t) + 3t \sin(\ln t) + \cos(\ln t)]/t^4, \qquad t \geq 1,$$

satisfies all the conditions of both of our theorems except that $r(t)$ changes signs. This equation has the oscillatory solution $x(t) = [\cos(\ln t)]/t$.

Remark. Suppose that $r(t) \leq 0$ and (iv) holds for $x \leq 0$. Then Theorem 1 would guarantee that no solution of (2) is oscillatory or nonpositive Z-type. The statement of Theorem 2 would remain the same but its proof would consist of ruling out the nonnegative Z-type solutions.

The proof of the following theorem is similar to the proofs above and will be omitted.

Theorem 3. *Suppose* $q(t) \geq 0$, (ii)–(iv) *hold,* $h: R \to R$ *is continuous, and either*

(v) $h(s) \leq 1$ *and* (i) *holds, or*

(vi) *for some constant* $d > 0$, $a(t) \geq d \geq h(s)$ *and* $x'' + q(t)x = 0$ *is nonoscillatory.*

Then no solution of

$$(a(t)x')' + q(t)f(x)h(x') = r(t) \tag{3}$$

is oscillatory or nonnegative Z-type. If, in addition, $h(s) \geq 0$ *for all* s, *then all solutions of* (3) *are nonoscillatory.*

The remark following Example 2 also applies to Theorem 3.

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