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ON A METHOD OF SOLVING THE INITIAL VALUE PROBLEM FOR THE WAVE EQUATION

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ABSTRACT. The wave equation is formally reduced to Laplace's equation by the change of variable $x_0 = it$, where $i = \sqrt{-1}$. In this paper we shall derive the well-known formula for the solution of Cauchy's problem of the wave equation from the integral representations of the solutions of Dirichlet and Neumann problems of Laplace's equation in the half-plane. Our method can be viewed as a hyperfunction-theoretic approach.

1. Introduction. The wave equation

$$\text{(W)} \qquad \frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \qquad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

is derived from Laplace's equation

$$\text{(L)} \qquad \partial^2 u / \partial x_0^2 + \Delta u = 0$$

by the change of variable

$$(1.1) \qquad x_0 = it \qquad (i = \sqrt{-1}).$$

This fact suggests that some information about equation (W) can be interpreted from those about equation (L) through (1.1). In fact, P. R. Garabedian [1] obtained the expression of solution of Cauchy's problem

$$\text{(WC)} \qquad \begin{aligned} &u_{tt} - \Delta u = 0, \\ &u(0, x) = f(x), \qquad x = (x_1, \cdots, x_n), \\ &u_t(0, x) = g(x), \end{aligned}$$

using a representation of a solution of (L) where $x_0 = it$. More precisely, he starts with the well-known formula for a harmonic function $v(x_0, x_1, \cdots, x_n)$:

$$v(x_0, x_1, \cdots, x_n) = \frac{1}{(n-1)\sigma_n} \int_{\partial D} \left[v \frac{\partial}{\partial \nu} \frac{1}{r^{n-1}} - \frac{1}{r^{n-1}} \frac{\partial v}{\partial \nu} \right] d\sigma,$$

where σ_n is the surface area of the n-dimensional unit sphere, ∂D is a

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closed surface bounding a domain D which contains the point $(x_0, \cdots, x_n)$, $\partial/\partial\nu$ is the normal derivation and

$$r = \sqrt{(x_0 - \xi_0)^2 + \cdots + (x_n - \xi_n)^2}.$$

He puts $x_0 = it$ and then deforms the surface of integration down upon the n-dimensional analogue of a pair of discs and a torus containing the $(n-1)$-dimensional sphere with center at $(0, x_1, \cdots, x_n)$: $(x_1 - \xi_1)^2 + \cdots + (x_n - \xi_n)^2 = t^2$, $x_0 = 0$, and finally shrinks the torus around the sphere. See also [2].

However, since equation (L) can be solved in the half-space $x_0 > 0$ under the condition $u(0, x) = f(x)$ or the condition $u_{x_0}(0, x) = g(x)$, it seems natural to expect that the solution of problem (WC) is derived from the solutions of the Dirichlet problem

$$\text{(LD)} \qquad u_{x_0x_0} + \Delta u = 0, \qquad u(0, x) = f(x),$$

and the Neumann problem

$$\text{(LN)} \qquad u_{x_0x_0} + \Delta u = 0, \qquad u_{x_0}(0, x) = g(x),$$

for the half-space $x_0 > 0$. The objective of the present paper is to show that the expectation is true.

The solutions of problems (LD) and (LN) are given explicitly by

$$(1.2)\quad \phi(x_0, x) = \frac{2x_0}{\sigma_n} \int \cdots \int_{R^n} \frac{f(\xi_1, \cdots, \xi_n)\, d\xi_1 \cdots d\xi_n}{(x_0^2 + (\xi_1 - x_1)^2 + \cdots + (\xi_n - x_n)^2)^{(n+1)/2}}$$

and

$$(1.3)\quad \psi(x_0, x) = \frac{-2}{(n-1)\sigma_n} \int \cdots \int_{R^n} \frac{g(\xi_1, \cdots, \xi_n)\, d\xi_1 \cdots d\xi_n}{(x_0^2 + (\xi_1 - x_1)^2 + \cdots + (\xi_n - x_n)^2)^{(n-1)/2}},$$

where σ_n is the area of the unit sphere of dimension n. On the other hand, the solution of problem (WC) is expressible as follows:

$$(1.4)\quad u(t, x) = \frac{1}{(n-2)!!} \left\{ \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-3)/2} (t^{n-2} \mu[f; t, x]) + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-3)/2} (t^{n-2} \mu[g; t, x]) \right\}$$

if $n \geq 3$ is odd,

$$u(t, x) = \frac{1}{(n-2)!!} \left\{ \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2} \int_0^t \mu[f; \rho, x] \frac{\rho^{n-1}\, d\rho}{\sqrt{t^2-\rho^2}} \right.$$

$$\left. + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2} \int_0^t \mu[g; \rho, x] \frac{\rho^{n-1}\, d\rho}{\sqrt{t^2-\rho^2}} \right\}, \tag{1.5}$$

if $n \geq 2$ is even, where

$$(n-2)!! = \begin{cases} (n-2)(n-4)\cdots\cdot 3\cdot 1, & n \text{ odd}, \\ (n-2)(n-4)\cdots\cdot 4\cdot 2, & n \text{ even}, \end{cases}$$

and

$$\mu[f; t, x] = \frac{1}{\sigma_{n-1}} \int_{\|\xi\|=1} \cdots \int f(x_1 + t\xi_1, \cdots, x_n + t\xi_n)\, d\omega_{n-1},$$

$d\omega_{n-1}$ being the area-element of the unit sphere of dimension $n-1$. *We shall derive* (1.4) *and* (1.5) *from* (1.2) *and* (1.3). Our method will be precisely stated in §2. We have tacitly made suitable assumptions on smoothness and integrability for f and g. These assumptions will be made throughout this paper.

2. **Preliminary considerations.** Consider the solutions (1.2) and (1.3). The variables $x_0, x_1, \cdots, x_n$ are usually assumed to be real: $(x_0, x) \in \mathbf{R}^{n+1}$. However, both ϕ and ψ are naturally extended into the complex domain by direct substitution of complex values of the variables $x_0, x_1, \cdots, x_n$. In fact, if given a point $(x_0, x_1, \cdots, x_n) \in \mathbf{C}^{n+1}$, we have

$$r^2 = x_0^2 + (\xi_1 - x_1)^2 + \cdots + (\xi_n - x_n)^2 \neq 0 \quad \text{for all } \xi \in \mathbf{R}^n, \tag{2.1}$$

then both ϕ and ψ are well defined at $(x_0, x_1, \cdots, x_n)$. For our purpose it is sufficient to limit ourselves to the special case where $x_1, \cdots, x_n$ remain real, while $x_0 \in \mathbf{C}$. Put $x_0 = s + it$. It is easy to see that (2.1) holds if $s > 0$ or $s < 0$. This implies that both ϕ and ψ are defined in the two half-spaces in $\mathbf{C} \times \mathbf{R}^n$: $\{(s + it, x);\ s > 0\}$ and $\{(-s + it, x);\ s > 0\}$. Moreover both ϕ and ψ are holomorphic with respect to x_0 in each of the half-spaces.

We remark that for a given $(it, x) \in \mathbf{C} \times \mathbf{R}^n$, $t > 0$, the set $\{(\xi_1, \cdots, \xi_n) \in \mathbf{R}^n;\ r = 0\}$ is the $(n-1)$-dimensional sphere with center at $(x_1, \cdots, x_n)$ and radius t.

It follows from the above considerations that

$$\phi(s + it, x), \qquad \phi(-s + it, x),$$
$$\psi(s + it, x), \qquad \psi(-s + it, x)$$

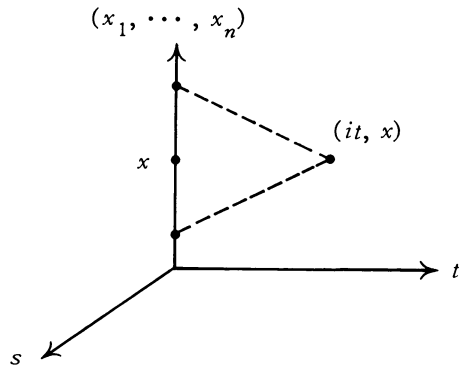

Figure 1

are well defined, provided $s > 0$. Since ϕ and ψ are holomorphic in x_0, we have

$$\frac{\partial\phi}{\partial s} = \frac{1}{i}\frac{\partial\phi}{\partial t}, \qquad \frac{\partial\psi}{\partial s} = \frac{1}{i}\frac{\partial\psi}{\partial t}. \tag{2.2}$$

From the representation of ϕ we have

$$\phi(s, x) = -\phi(-s, x), \tag{2.3}$$

so

$$\partial\phi(s, x)/\partial s = \partial\phi(-s, x)/\partial s. \tag{2.4}$$

Since $\phi(s, x) \to f(x)$ as $s \to 0$, we have from (2.3)

$$\tfrac{1}{2}(\phi(s, x) - \phi(-s, x)) \to f(x) \qquad (s \to 0).$$

On the other hand, we have from (2.2) and (2.4)

$$\tfrac{1}{2}(\partial\phi(s, x)/\partial t - \partial\phi(-s, x)/\partial t) \to 0 \qquad (s \to 0).$$

One may expect from the above limits that

$$\Phi(s, t, x) = \tfrac{1}{2}(\phi(s + it, x) - \phi(-s + it, x))$$

does converge to the solution of Cauchy's problem

$$u_{tt} - \Delta u = 0, \qquad u(0, x) = f(x), \qquad u_t(0, x) = 0 \tag{2.5}$$

as $s \to 0$. We shall prove, in the following sections, that this expectation is true.

Next consider ψ. We have $\psi(s, x) = \psi(-s, x)$ and hence $\partial\psi(s, x)/\partial s = -\partial\psi(-s, x)/\partial s$. Observing that $\partial\psi(s, x)/\partial s \to g(x)$ $(s \to 0)$, we have from (2.2)

$$\frac{1}{2i}\left(\frac{\partial\psi}{\partial t}(s, x) - \frac{\partial\psi}{\partial t}(-s, x)\right) \to g(x) \qquad (s \to 0).$$

On the other hand,

$$(1/2i)(\psi(s, x) - \psi(-s, x)) \to 0 \qquad (s \to 0).$$

Therefore it may be expected that

$$\Psi(s, t, x) = (1/2i)(\psi(s + it, x) - \psi(-s + it, x))$$

does converge to the solution of the problem

$$u_{tt} - \Delta u = 0, \qquad u(0, x) = 0, \qquad u_t(0, x) = g(x) \tag{2.6}$$

as $s \to 0$. We shall also prove that this expectation is true. It is evident that the solution of problem (WC) is the sum of the solution of (2.5) and (2.6).

It should be remarked that the solutions $\phi(x_0, x)$ and $\psi(x_0, x)$ are written in terms of $\mu[f; t, x]$ and $\mu[g; t, x]$ as

$$\phi(x_0, x) = \frac{2\sigma_{n-1}}{\sigma_n} x_0 \int_0^\infty \mu[f; \rho, x] \frac{\rho^{n-1}\, d\rho}{(x_0^2 + \rho^2)^{(n+1)/2}} \tag{2.7}$$

and

$$\psi(x_0, x) = \frac{-2\sigma_{n-1}}{(n-1)\sigma_n} \int_0^\infty \mu[g; \rho, x] \frac{\rho^{n-1}\, d\rho}{(x_0^2 + \rho^2)^{(n-1)/2}}. \tag{2.8}$$

Simple calculations show

$$\sigma_{n-1}/\sigma_n = \begin{cases} ((n-1)!!/(n-2)!!) \cdot 1/\pi, & n \geq 3 \text{ odd}, \\ ((n-1)!!/(n-2)!!) \cdot \frac{1}{2}, & n \geq 2 \text{ even}. \end{cases}$$

3. **Calculation of Φ and Ψ.** The first step of our procedure is to calculate integrals of the form

$$I = \int_0^\infty F(\rho) \frac{\rho^{n-1}\, d\rho}{(x_0^2 + \rho^2)^{(n+1)/2}} \tag{3.1}$$

and

$$J = \int_0^\infty F(\rho) \frac{\rho^{n-1}\, d\rho}{(x_0^2 + \rho^2)^{(n-1)/2}}. \tag{3.2}$$

First consider the integral I, which is written as

$$I = \frac{-1}{n-1} \int_0^\infty \rho^{n-2} F(\rho)\, d\left(\frac{1}{(x_0^2 + \rho^2)^{(n-1)/2}}\right).$$

We have by integration by parts

$$I = \frac{-1}{n-1}\left[\frac{\rho^{n-2}F(\rho)}{(x_0^2+\rho^2)^{(n-1)/2}}\right]_0^\infty + \frac{1}{n-1}\int_0^\infty \frac{d}{d\rho}(\rho^{n-2}F(\rho))\,\frac{d\rho}{(x_0^2+\rho^2)^{(n-1)/2}}$$

under suitable smoothness and growth conditions on $F(\rho)$. Suppose that the first term on the right-hand side vanishes:

$$[\rho^{n-2}F(\rho)/(x_0^2+\rho^2)^{(n-1)/2}]_0^\infty = 0,$$

which holds under a suitable condition on F. We thus have

$$I = \frac{1}{n-1}\int_0^\infty \frac{1}{\rho}\frac{d}{d\rho}(\rho^{n-2}F(\rho))\,\frac{\rho\,d\rho}{(x_0^2+\rho^2)^{(n-1)/2}}.$$

The integration by parts can be repeated under suitable conditions to obtain

$$I = \begin{cases} \dfrac{1}{(n-1)!!}\displaystyle\int_0^\infty \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{(n-1)/2}(\rho^{n-2}F(\rho))\,\frac{\rho\,d\rho}{x_0^2+\rho^2}, & n \text{ odd}, \\ \dfrac{1}{(n-1)!!}\displaystyle\int_0^\infty \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{n/2}(\rho^{n-2}F(\rho))\,\frac{\rho\,d\rho}{\sqrt{x_0^2+\rho^2}}, & n \text{ even}. \end{cases}$$

We proceed to the integral (3.2). The integration by parts leads us to

$$J = \frac{-1}{n-3}\left[\frac{\rho^{n-2}F(\rho)}{(x_n^2+\rho^2)^{(n-3)/2}}\right]_0^\infty + \frac{1}{n-3}\int_0^\infty \frac{1}{\rho}\frac{d}{d\rho}(\rho^{n-2}F(\rho))\,\frac{\rho d\rho}{(x_0^2+\rho^2)^{(n-3)/2}}.$$

Supposing that the first term on the right-hand side vanishes, we obtain

$$J = \frac{1}{n-3}\int_0^\infty \frac{1}{\rho}\frac{d}{d\rho}(\rho^{n-2}F(\rho))\,\frac{\rho d\rho}{(x_0^2+\rho^2)^{(n-3)/2}}.$$

By repeatedly integrating by parts under similar conditions, we arrive at

$$J = \begin{cases} \dfrac{1}{(n-3)!!}\displaystyle\int_0^\infty \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{(n-3)/2}(\rho^{n-2}F(\rho))\,\frac{\rho d\rho}{x_0^2+\rho^2}, & n \text{ odd}, \\ \dfrac{1}{(n-3)!!}\displaystyle\int_0^\infty \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{(n-2)/2}(\rho^{n-2}F(\rho))\,\frac{\rho d\rho}{\sqrt{x_0^2+\rho^2}}, & n \text{ even}. \end{cases}$$

The above calculations can be applied to (2.7) and (2.8), if f and g satisfy conditions usually made for problem (WC). We thus obtain

$$\Phi(s, t, x) = \begin{cases} \dfrac{1}{(n-2)!! \cdot \pi} \displaystyle\int_0^\infty \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{(n-1)/2} (\rho^{n-2}\mu[f; \rho, x]) d_0(s, t, \rho)\rho\, d\rho, \\[2ex] \dfrac{1}{(n-2)!! \cdot 2} \displaystyle\int_0^\infty \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{n/2} (\rho^{n-2}\mu[f; \rho, x]) d_e(s, t, \rho)\rho\, d\rho \end{cases}$$

and

$$\Psi(s, t, x) = \begin{cases} \dfrac{-1}{(n-2)!! \cdot \pi i} \displaystyle\int_0^\infty \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{(n-3)/2} (\rho^{n-2}\mu[g; \rho, x]) n_0(s, t, \rho)\rho\, d\rho, \\[2ex] \dfrac{-1}{(n-2)!! \cdot 2i} \displaystyle\int_0^\infty \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{(n-2)/2} (\rho^{n-2}\mu[g; \rho, x]) n_e(s, t, \rho)\rho\, d\rho \end{cases}$$

according as n is odd or even, where

$$d_0(s, t, \rho) = \frac{s + it}{(s + it)^2 + \rho^2} - \frac{-s + it}{(-s + it)^2 + \rho^2},$$

$$d_e(s, t, \rho) = \frac{s + it}{\sqrt{(s + it)^2 + \rho^2}} - \frac{-s + it}{\sqrt{(-s + it)^2 + \rho^2}},$$

$$n_0(s, t, \rho) = \frac{1}{(s + it)^2 + \rho^2} - \frac{1}{(-s + it)^2 + \rho^2},$$

$$n_e(s, t, \rho) = \frac{1}{\sqrt{(s + it)^2 + \rho^2}} - \frac{1}{\sqrt{(-s + it)^2 + \rho^2}}.$$

Our problem is therefore reduced to computing the limits of integrals of the form

$$D_0(s, t) = \int_0^\infty F(\rho) d_0(s, t, \rho)\rho\, d\rho, \qquad D_e(s, t) = \int_0^\infty F(\rho) d_e(s, t, \rho)\rho\, d\rho,$$

$$N_0(s, t) = \int_0^\infty F(\rho) n_0(s, t, \rho)\rho\, d\rho, \qquad N_e(s, t) = \int_0^\infty F(\rho) n_e(s, t, \rho)\rho\, d\rho$$

as $s \to 0$.

4. **Completion of the proof.** Suppose that $t > 0$ and $s > 0$. First consider the integral $N_0(s, t)$. Since

$$n_0(s, t, \rho) = -4ist/((\rho^2 - t^2 + s^2)^2 + (2st)^2),$$

$N_0(s, t)$ is written as

$$N_0(s, t) = -i \int_0^\infty F(\rho) \frac{2st \cdot 2\rho\, d\rho}{(\rho^2 - t^2 + s^2)^2 + (2st)^2}.$$

Taking a small positive number δ, divide $N_0(s, t)$ into three parts:

$$N_0(s, t) = I_1(s, t) + I_2(s, t) + I_3(s, t)$$

with $I_1(s, t) = -i\int_0^{t-\delta}$, $I_2(s, t) = -i\int_{t-\delta}^{t+\delta}$ and $I_3(s, t) = -i\int_{t+\delta}^\infty$. It is easy to see that $I_1(s, t)$ and $I_3(s, t)$ tend to zero as $s \to 0$. Consider $I_2(s, t)$. If we put $F(\rho) = F(t) + \epsilon(\rho)$ then $\max\{|\epsilon(\rho)|;\ \rho \in [t-\delta, t+\delta]\} \to 0$ as $\delta \to 0$. Observing that

$$\int_{t-\delta}^{t+\delta} \frac{2st \cdot 2\rho\, d\rho}{(\rho^2 - t^2 + s^2)^2 + (2st)^2} = \left[\arctan \frac{\rho^2 - t^2 + s^2}{2st}\right]_{t-\delta}^{t+\delta} \to \pi \quad (s \to 0)$$

we see that

$$I_2(s, t) = -i \int_{t-\delta}^{t+\delta} (F(t) + \epsilon(\rho)) \frac{2st \cdot 2\rho\, d\rho}{(\rho^2 - t^2 + s^2)^2 + (2st)^2}$$

converges to $-\pi i F(t) + \eta(\delta)$ as $s \to 0$, where $\eta(\delta) \to 0$ $(\delta \to 0)$.

We thus get

$$N_0(s, t) \to -\pi i F(t) \qquad (s \to 0). \tag{4.1}$$

We conclude from this that if $n \geq 3$ is odd, then

$$\Psi(s, t, x) \to \frac{1}{(n-2)!!} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} (t^{n-2}\mu[g;\ t, x]) \tag{4.2}$$

as $s \to 0$.

Next consider the integral $D_0(s, t)$, written as

$$D_0(s, t) = s \int_0^\infty F(\rho) \left(\frac{1}{(s+it)^2 + \rho^2} + \frac{1}{(-s+it)^2 + \rho^2}\right) \rho\, d\rho.$$

$$+ it \int_0^\infty F(\rho) n_0(s, t, \rho) \rho\, d\rho.$$

Under a suitable growth condition on F, it is easy to see that the first term on the right-hand side tends to zero as $s \to 0$. It follows from this and (4.1) that $D_0(s, t) - \pi t F(t)$ $(s \to 0)$. We thus conclude that if $n \geq 3$ is odd, then

$$\lim_{s \to 0} \Phi(s, t, x) = \frac{1}{(n-2)!!} \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} (t^{n-2}\mu[f;\ t, x]). \tag{4.3}$$

Combining (4.2) and (4.3) we obtain the expression (1.4).

Before we proceed to the integrals $D_e(s, t)$ and $N_e(s, t)$, we have to

specify the branches used here of the doubly-valued functions $\sqrt{(s+it)^2+\rho^2}$ and $\sqrt{(-s+it)^2+\rho^2}$. The function $\sqrt{x_0^2+\rho^2}$ has as its branch loci in $\mathbf{C}\times\mathbf{R}$ the two lines $x_0 = \pm i\rho$.

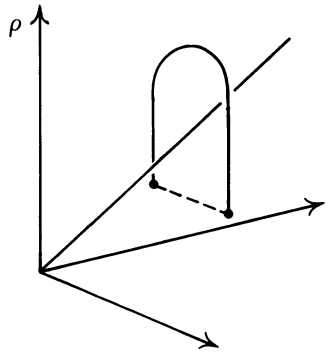

Figure 2

It seems natural to use the branches of $\sqrt{(s+it)^2+\rho^2}$ and $\sqrt{(-s+it)^2+\rho^2}$ determined by the following conditions:

$$\lim_{s\to 0}\sqrt{(s+it)^2+\rho^2} = \lim_{s\to 0}\sqrt{(-s+it)^2+\rho^2} = \sqrt{\rho^2-t^2} \quad \text{for } 0<t<\rho$$

and

$$\lim_{s\to 0}\sqrt{(s+it)^2+\rho^2} = -\lim_{s\to 0}\sqrt{(-s+it)^2+\rho^2} = i\sqrt{t^2-\rho^2} \quad \text{for } 0<\rho<t,$$

(see Figure 2). From this convention it follows that $n_e(s, t, \rho)$ tends to $2/i\sqrt{t^2+\rho^2}$ for $0<\rho<t$ and tends to zero for $0<t<\rho$ as $s\to 0$.

Consider the integral $N_e(s, t)$, which we write as

$$N_e(s, t) = \int_0^t F(\rho)n_e(s, t, \rho)\rho\, d\rho + \int_t^\infty F(\rho)n_e(s, t, \rho)\rho\, d\rho.$$

Under suitable conditions on F, we have

$$\int_0^t F(\rho)n_e(s, t, \rho)\rho\, d\rho \to \frac{2}{i}\int_0^t F(\rho)\frac{\rho\, d\rho}{\sqrt{t^2-\rho^2}}$$

and $\int_t^\infty F(\rho)n_e(s, t, \rho)\rho\, d\rho \to 0$ as $s\to 0$, from which

$$N_e(s, t) \to \frac{2}{i}\int_0^t F(\rho)\frac{\rho\, d\rho}{\sqrt{t^2-\rho^2}} \qquad (s\to 0). \tag{4.4}$$

Applying this result to $\Psi(s, t, x)$, we get

$$\lim_{s\to 0} \Psi(s, t, x) = \frac{1}{(n-2)!!} \int_0^t \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^{(n-2)/2} (\rho^{n-2}\mu[f; \rho, x]) \frac{\rho\, d\rho}{\sqrt{t^2-\rho^2}}$$

if $n \geq 2$ is even. Utilizing $(n-2)/2$ times the following formula:

$$\int_0^t \frac{1}{\rho}\frac{d}{d\rho} F(\rho) \frac{\rho\, d\rho}{\sqrt{t^2-\rho^2}} = \frac{1}{t}\frac{d}{dt}\int_0^t F(\rho) \frac{\rho\, d\rho}{\sqrt{t^2-\rho^2}},$$

which is valid for $F \in C^1$ with $F(0) = 0$, we can rewrite the above limit as

$$\text{(4.5)} \quad \lim_{s\to 0} \Psi(s, t, x) = \frac{1}{(n-2)!!} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2} \int_0^t \mu[g; \rho, x] \frac{\rho^{n-1}\, d\rho}{\sqrt{t^2-\rho^2}}.$$

Finally consider the integral $D_e(s, t)$. Since

$$d_e(s, t, \rho) = s\left(\frac{1}{\sqrt{(s+it)^2+\rho^2}} + \frac{1}{\sqrt{(-s+it)^2+\rho^2}}\right) + itn_e(s, t, \rho),$$

we have

$$D_e(s, t) = s\int_0^\infty F(s)\left(\frac{1}{\sqrt{(s+it)^2+\rho^2}} + \frac{1}{\sqrt{(-s+it)^2+\rho^2}}\right)\rho\, d\rho$$

$$+ it\int_0^\infty F(s) n_e(s, t, \rho)\rho\, d\rho.$$

It is easy to see that the first term on the right-hand side tends to zero as $s \to 0$, if F satisfies suitable conditions. Then we have from (4.4)

$$\lim_{s\to 0} D_e(s, t) = 2t\int_0^t F(\rho) \frac{\rho\, d\rho}{\sqrt{t^2-\rho^2}},$$

whence we have

$$\text{(4.6)} \quad \lim_{s\to 0} \Phi(s, t, x) = \frac{1}{(n-2)!!} \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2} \int_0^t \mu[f; \rho, x] \frac{\rho^{n-1}\, d\rho}{\sqrt{t^2-\rho^2}}$$

for even $n \geq 2$.

The relations (4.5) and (4.6) prove (1.5).

We make the following remark: *Huygens' principle is valid for the wave equation when the function*

$$(x_0^2 + (x_1-\xi_1)^2 + \cdots + (x_n-\xi_n)^2)^{-(n-1)/2}$$

is single-valued.

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A BOUND-TWO ISOMORPHISM BETWEEN $C(X)$ BANACH SPACES

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ABSTRACT. Nonhomeomorphic compact Hausdorff spaces X and Y and an isomorphism $\phi: C(X) \to C(Y)$ (onto) are constructed such that $\|\phi\| \, \|\phi^{-1}\| = 2$. Amir had asked if such a ϕ exists with $\|\phi\| \, \|\phi^{-1}\| < 3$.

1. **Introduction.** In 1965, D. Amir [1] showed that if $C(X)$ and $C(Y)$ admit an (onto) isomorphism ϕ whose bound $\|\phi\| \, \|\phi^{-1}\|$ is less than 2, then X and Y are homeomorphic. Here X and Y are compact Hausdorff spaces and $C(X)$, $C(Y)$ are the sup-norm Banach spaces of real-valued continuous functions on X and Y, respectively. In the same paper he constructed nonhomeomorphic X and Y and an isomorphism ϕ: $C(X) \to C(Y)$ with bound exactly 3. Amir posed the problem of determining whether or not an isomorphism can exist for nonhomeomorphic X and Y with $\|\phi\| \, \|\phi^{-1}\| < 3$. See also [5, p. 155]. The purpose of this paper is to exhibit such an example, with $\|\phi\| \, \|\phi^{-1}\| = 2$.

We note that a similar problem was settled by Camburn for the larger class of sup-norm Banach spaces $C_0(T)$ of complex valued continuous

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