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ON A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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ABSTRACT. Consider a bounded domain $G \subset R^N$ $(N \geq 1)$ with smooth boundary Γ. Let L be a uniformly elliptic linear differential operator. Let γ and β be two maximal monotone mappings in R. We prove that, when γ satisfies a certain growth condition, given $f \in L^2(G)$ there is $u \in H^2(G)$ such that

$$Lu + \gamma(u) \ni f \quad \text{a.e. on } G, \qquad \text{and} \qquad -\partial u/\partial \nu \in \beta(u|_\Gamma) \quad \text{a.e. on } \Gamma,$$

where $\partial u/\partial \nu$ is the conormal derivative associated with L.

1. Let $G \subset R^N$ $(N \geq 1)$ be a bounded domain with smooth boundary Γ. Consider the uniformly elliptic linear operator

$$Lu = -D_j(a_{ij}(x)D_iu) + b_i(x)D_iu + c(x)u,$$

$$a_{ij} = a_{ji} \in C^1(\overline{G}); \; b_i, c \in L^\infty(G) \qquad (i, j = 1, 2, \cdots, N),$$

$$a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \quad c > 0 \text{ constant}, \; \forall x \in G, \; \xi \in R^N.$$

(All functions and scalars that we consider are real.)

Let $\gamma: R \to 2^R$ be a maximal monotone mapping. The domain $D(\gamma)$ of γ is the set of all numbers s such that $\gamma(s) \neq \emptyset$. For each $s \in D(\gamma)$, $\gamma(s)$ is a closed interval and thus contains a unique element, which we denote by $\gamma^0(s)$, having smallest absolute value. We assume that the mapping γ satisfies the condition

$$(1) \qquad |\gamma^0(s)| \geq \phi(s)|s|, \quad \forall s \in D(\gamma) \text{ with } \lim_{|s| \to \infty} \phi(s) = \infty.$$

It can be verified that γ induces a maximal monotone mapping $\overline{\gamma}: L^2(G) \to 2^{L^2(G)}$ in a natural way:

$$\overline{\gamma}(u) = \{v \in L^2(G) | v(x) \in \gamma(u(x)) \text{ a.e.}\} \qquad (u \in L^2(G)).$$

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Similarly, let $\beta: R \to 2^R$ be another maximal monotone mapping and let $\bar{\beta}$ be the mapping in $L^2(\Gamma)$ induced by β.

Proposition. *Suppose that all the conditions on the operator* L *and the mappings* γ *and* β *described above are satisfied. Suppose also that there exists* $s_0 \in D(\gamma) \cap D(\beta)$ *with* $0 \in \beta(s_0)$. *Then for every given* $f \in L^2(G)$, *there exists* $u \in H^2(G)$ *such that*

$$Lu + \bar{\gamma}(u) \ni f \text{ in the sense of } L^2(G), \tag{2}$$

$$-\partial u/\partial \nu = a_{ij} D_i u \cdot \cos(n, x_j) \in \bar{\beta}(u|_\Gamma), \tag{3}$$

where n *is the outward normal to* Γ.

Before proving this proposition in §2, we would like to make a few comments. When (i) $\gamma \equiv 0$ and the bilinear form

$$a(u, v) = \int_G (a_{ij} D_i u D_j v + b_i D_i u \cdot v + cuv)\, dx$$

is coercive on $H^1(G)$, or (ii) $Lu = -\Delta u + u$, the proposition has been proved by H. Brézis in [3] and [2] respectively. It seems to us that the general case, where no coercivity is assumed, cannot be immediately reduced to these cases.

The corresponding Dirichlet problem

$$Lu + \bar{\gamma}(u) \ni f, \tag{4}$$

$$u|_\Gamma = 0, \tag{5}$$

has been studied by P. Hess [7], and condition (1) on γ is similar to his condition in [7]. Our method can also be applied to this Dirichlet problem, and the argument will be simpler than the Neumann-type problem considered here. We feel that the main difference between our method as applied to the Dirichlet problem and that of P. Hess lies in proving the existence of a solution for the approximate equation. He uses a theorem on the solvability of a functional equation involving a demicontinuous mapping of type (S^+) which is similar to one established by him earlier [6, Theorem 1], using a homotopy argument. Instead, we shall use the well-known Schauder fixed point theorem (see e.g. [4, p. 105]).

2. The proof makes use of the concept of the Yosida approximation. Let $U: H \to 2^H$ be a maximal monotone operator in a Hilbert space H. Then for every $\epsilon > 0$, $(I + \epsilon U)^{-1}$ is a nonexpansive mapping defined on all of H. The mapping

$$U_\epsilon = [I - (I + \epsilon U)^{-1}]/\epsilon \tag{6}$$

is called the Yosida approximation of U (at ϵ). U_ϵ is Lipschitzian with Lipschitz constant $2/\epsilon$ and monotone. For more details on Yosida approximations we refer the reader to M. Crandall and A. Pazy [5], or T. Kato [8]. In the case of the maximal monotone mappings γ and β introduced in §1, it can be verified that the Yosida approximation of $\bar{\gamma}$ for example is generated by γ_ϵ.

Proof of the proposition. We observe that by shifting and changing variable, we can assume without loss of generality that $0 \in \gamma(0)$ and $0 \in \beta(0)$. Then $\gamma_\epsilon(0) = \beta_\epsilon(0) = 0$. The proof consists of proving that the approximate problem

$$Lu + \bar{\gamma}_\epsilon(u) = f, \tag{7}$$

$$-\partial u/\partial \nu = \bar{\beta}_\epsilon(u_{|\Gamma}) \tag{8}$$

has a solution $u_\epsilon \in H^2(G)$ for all $\epsilon > 0$ sufficiently small. We then pass to the limit as $\epsilon \downarrow 0$ using estimates for u_ϵ independent of ϵ.

I. *Proof that the approximate problem has a solution.* Let

$$L'u = -D_j(a_{ij}D_iu), \qquad L'' = L - L',$$

$$a'(u, v) = \int_G a_{ij}D_iuD_jv\,dx \qquad (u, v \in H^1(G)).$$

For u given in $H^1(G)$, the linear form

$$w \longrightarrow a'(u, w) + \int_G [\gamma_\epsilon(u) + u]w\,dx + \int_\Gamma \beta_\epsilon(u_{|\Gamma})w\,d\Gamma \qquad (w \in H^1(G)),$$

is continuous on $H^1(G)$, so that there is an element $\mathfrak{A}u \in [H^1(G)]'$ with

$$\langle \mathfrak{A}u, w\rangle = a'(u, w) + \int_G [\gamma_\epsilon(u) + u]w\,dx + \int_\Gamma \beta_\epsilon(u_{|\Gamma})w\,d\Gamma \qquad (w \in H^1(G)).$$

It can be verified that the mapping $u \longrightarrow \mathfrak{A}u$ is bounded, hemicontinuous, strictly monotone and coercive. Therefore (see [4, Theorem 1] or [9, Chapter 2, Theorem 2.1]) for every given $v \in H^1(G)$ there exists a unique $u \in H^1(G)$ such that for all $w \in H^1(G)$

$$\langle \mathfrak{A}u, w\rangle = a'(u, w) + \int_G [\gamma_\epsilon(u) + u]w\,dx + \int_\Gamma \beta_\epsilon(u_{|\Gamma})w\,d\Gamma$$
$$= \int_G (f - L''v + v)w\,dx.$$

We then deduce that the boundary value problem

$$L'u + \gamma_\epsilon(u) + u = f - L''v + v \text{ in the sense of } \mathfrak{D}'(G), \tag{9}$$

$$-\partial u/\partial \nu = \beta_\epsilon(u_{|\Gamma}) \tag{10}$$

has a unique solution $u_v \in H^1(G)$. If we bring $\gamma_\epsilon(u)$ to the right-hand side in equation (9), then it follows from [3, Theorem I. 10], that $u_v \in H^2(G)$.

It now suffices to show that the mapping $T_\epsilon\colon v \to u_v$ in $H^1(G)$ has a fixed point.

(a) *The mapping* T_ϵ *is continuous.* Let $v_1, v_2 \in H^1(G)$ and $u_1 = T_\epsilon(v_1)$, $u_2 = T_\epsilon(v_2)$. The continuity of T_ϵ can be seen by taking the difference of equations (9) corresponding to v_1 and v_2 and then taking the inner product in $L^2(G)$ of this with $u_1 - u_2$.

(b) *There is an integer* $K > 0$ *and an* $\epsilon_0 > 0$ *such that* T_ϵ $(0 < \epsilon < \epsilon_0)$ *maps the closed ball* $B_K(0)$ *of* $H^1(G)$ *into itself.* If not, for each $n = 1, 2, \cdots$ there are ϵ_n with $0 < \epsilon_n < n^{-1}$, v_n with $\|v_n\|_1 \le n$, u_n with $\|u_n\|_1 > n$ (where $\|\cdot\|_1$ is the norm in $H^1(G)$) satisfying (9) and (10) with $\epsilon = \epsilon_n$. Taking the inner product in $L^2(G)$ of (9) with $n^{-1}w_n = n^{-2}u_n$ we obtain

$$a'(w_n, w_n) + \int_\Gamma n^{-2}\beta_{\epsilon_n}(u_n)u_n \, d\Gamma + \int_G n^{-2}\gamma_{\epsilon_n}(u_n)u_n \, dx + \|w_n\|_0^2$$
$$= \|n^{-1}(f - L''v_n + v_n)\|_0 \cdot \|w_n\|_0,$$

where $\|\cdot\|_0$ is the norm in $L^2(G)$. From this we deduce

$$(11) \qquad 1 < \|w_n\|_1 < C \qquad (n = 1, 2, \cdots),$$

$$(12) \qquad \int_G n^{-2}\gamma_{\epsilon_n}(u_n)u_n \, dx < C \qquad (n = 1, 2, \cdots).$$

Here and in the sequel C denotes various positive constants independent of γ, β, ϵ. Now taking the inner product in $L^2(G)$ of (9) with $n^{-2}\gamma_{\epsilon_n}(u_n)$ we obtain (for justification see [1] or [10, Appendix I] for a special case), recalling $w_n = n^{-1}u_n$:

$$\int_G \gamma'_{\epsilon_n}(u_n)a_{ij}D_iw_nD_jw_n \, dx + \int_\Gamma n^{-2}\beta_{\epsilon_n}(u_n)\gamma_{\epsilon_n}(u_n) \, d\Gamma + \int_G n^{-2}\gamma_{\epsilon_n}(u_n)u_n \, dx$$
$$+ \|n^{-1}\gamma_{\epsilon_n}(u_n)\|_0^2 \le \|n^{-1}(f - L''v_n + v_n)\|_0 \cdot \|n^{-1}\gamma_{\epsilon_n}(u_n)\|_0.$$

The third integral is nonnegative. Since $0 \in \gamma(0)$, $\gamma_{\epsilon_n}(\cdot)$ is monotone increasing so that $\gamma'_{\epsilon_n}(u_n(x)) \ge 0$ a.e. and the first integral is also nonnegative. Moreover, we observe that if $u_n(x) = 0$ then $\beta_{\epsilon_n}(u_n)\gamma_{\epsilon_n}(u_n) = 0$, and if $u_n(x) \ne 0$ then

$$\beta_{\epsilon_n}(u_n)\gamma_{\epsilon_n}(u_n) = \beta_{\epsilon_n}(u_n)u_n \cdot \gamma_{\epsilon_n}(u_n)u_n \cdot u_n^{-2} \ge 0,$$

so that the second integral is also nonnegative. We then deduce that

$$\|n^{-1}\gamma_{\epsilon_n}(u_n)\|_0 < C \qquad (n = 1, 2, \cdots).$$

We now write

$$L'w_n + w_n = n^{-1}(f - L''v_n + v_n - \gamma_{\epsilon_n}(u_n)), \tag{13}$$

$$-\partial w_n/\partial\nu = n^{-1}\beta_{\epsilon_n}(u_n|_\Gamma). \tag{14}$$

Since the right-hand side of (13) remains bounded in $L^2(G)$, it follows from [3, Theorem I.10], that $\|w_n\|_2 < C$ $(n = 1, 2, \cdots)$, where $\|\cdot\|_2$ denotes the norm in $H^2(G)$. Because the imbedding of $H^2(G)$ into $H^1(G)$ is compact, we can extract a subsequence, still denoted by $\{w_n\}$, of $\{w_n\}$ such that w_n converges strongly in $H^1(G)$ to w and w_n converges a.e. on G to w. Since $\|w\|_1 \geq 1$ $(n = 1, 2, \cdots)$, $w(x) \neq 0$ on a subset of G of nonzero measure. We shall see that this contradicts condition (1) on γ and (12). In fact, putting $s_n(x) = (I + \epsilon_n\gamma)^{-1}u_n(x)$, we obtain with $t_n(x) \in \gamma(s_n(x))$

$$u_n(x) = s_n(x) + \epsilon_n t_n(x), \tag{15}$$

$$n^{-2}\gamma_{\epsilon_n}(u_n(x))u_n(x) = n^{-2}u_n(x)t_n(x) = |w_n(x)| \cdot n^{-1}|t_n(x)|. \tag{16}$$

Consider $x \in G$ with $\lim_n |w_n(x)| > 0$, i.e. $\lim_n |u_n(x)| = \infty$. Then

$$\lim_n n^{-1}|t_n(x)| = \infty.$$

For otherwise there would be a subsequence such that

$$\sup_k n_k^{-1}|t_{n_k}(x)| < \infty.$$

From (15) it then follows that

$$\liminf_k n_k^{-1}|s_{n_k}(x)| \geq \lim_k n_k^{-1}|u_{n_k}(x)| > 0. \tag{17}$$

By condition (1) on γ,

$$n_k^{-1}|t_{n_k}(x)| \geq n_k^{-1}|\gamma^0(s_{n_k}(x))| > \phi(s_{n_k}(x))n_k^{-1}|s_{n_k}(x)|.$$

Since $\lim_k |s_{n_k}(x)| = \infty$, $\lim_k \phi(s_{n_k}(x)) = \infty$. This together with (17) shows that

$$\lim_k n_k^{-1}|t_{n_k}(x)| = \infty$$

and we thus arrive at a contradiction. From (16) we therefore see that

$$\lim_n n^{-2}\gamma_{\epsilon_n}(u_n(x))u_n(x) = \infty$$

on a subset of G of nonzero measure. By Fatou's lemma, this contradicts (12).

(c) *The mapping* T_ϵ $(0 < \epsilon < \epsilon_0)$ *of* $B_K(0)$ *into itself is relatively compact*. In fact, by an argument similar to that in the last step, we see that for all $v \in B_K(0)$, $\|T_\epsilon(v)\|_2 < C$. Since the imbedding of $H^2(G)$ into $H^1(G)$ is compact, we deduce that the closure of $T_\epsilon(B_K(0))$ is compact.

Thus by the Schauder fixed point theorem [4,p. 105], T_ϵ $(0 < \epsilon < \epsilon_0)$ has a fixed point in $B_K(0)$.

II. *Passing to the limit as* $\epsilon \downarrow 0$. Using the same argument as in Step I(b) above (take the inner product in $L^2(G)$ of (7) with u and then with $\overline{\gamma}_\epsilon(u)$), we see that there is a constant C independent of ϵ such that a solution u_ϵ of (7) and (8) satisfies

$$\|u_\epsilon\|_2 < C, \quad \|\overline{\gamma}_\epsilon(u_\epsilon)\|_0 < C \qquad (0 < \epsilon < \epsilon_0).$$

Since the mapping $u \longrightarrow u_{|\Gamma}$ of $H^1(G)$ onto $H^{1/2}(\Gamma) \subset L^2(\Gamma)$ is continuous, we can extract a subsequence $\{u_{\epsilon_n}\}$ with the following properties

u_{ϵ_n} converges weakly to u in $H^2(G)$,
Lu_{ϵ_n} converges weakly to Lu in $L^2(G)$,
u_{ϵ_n} converges strongly to u in $H^1(G)$,
$\overline{\gamma}_{\epsilon_n}(u_{\epsilon_n})$ converges weakly to $-Lu + f$ in $L^2(G)$,
$\partial u_{\epsilon_n}/\partial\nu$ converges weakly to $\partial u/\partial\nu$ in $L^2(\Gamma)$
(i.e. $\overline{\beta}_{\epsilon_n}(u_{\epsilon_n|\Gamma})$ converges weakly to $-\partial u/\partial\nu$ in $L^2(\Gamma)$),
$u_{\epsilon_n|\Gamma}$ converges strongly to $u_{|\Gamma}$ in $L^2(\Gamma)$.

From a property of Yosida approximations [8, Lemma 4.5], it then follows that

$$u \in D(\overline{\gamma}), \quad -Lu + f \in \overline{\gamma}(u); \qquad u_{|\Gamma} \in D(\overline{\beta}), \quad -\partial u/\partial\nu \in \overline{\beta}(u_{|\Gamma})$$

and the proof is complete.

From the proposition we deduce the following

Corollary. *Suppose that the conditions in the proposition are satisfied. Then for any* $k_1 \geq 0$, $k_2 > 0$, *the boundary value problem*

$$Lu + \overline{\gamma}(u) \ni f, \qquad -k_1 u - k_2\partial u/\partial\nu \in \overline{\beta}(u_{|\Gamma})$$

has a solution $u \in H^2(G)$.

Proof. The boundary condition can be written as

$$-\partial u/\partial\nu \in k_1 k_2^{-1} u + k_2^{-1}\beta(u_{|\Gamma}).$$

On the other hand, it can be verified that $k_1 k_2^{-1} I + k_2^{-1}\beta$ is a maximal monotone mapping in R, using the well-known fact that a monotone mapping U in a Hilbert space H is maximal if and only if for all $\lambda > 0$ the range of $I + \lambda U$ is the whole of H (see e.g. [2]).

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