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LOCAL EUCLIDEAN FOUR-POINT PROPERTIES WHICH CHARACTERIZE INNER-PRODUCT SPACES

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ABSTRACT. Let M be a complete, convex, externally convex metric space. We show M is an inner-product space if and only if for each point t of M, M contains a sphere S_t which has the euclidean queasy, feeble or weak four-point property.

1. **Introduction.** The purpose of this paper is to give generalizations of the Blumenthal and Day characterizations of inner-product spaces among the class of complete, convex, externally convex metric spaces. For definitions and a detailed study of these concepts, see [3]. A metric space M has the euclidean weak four-point property provided each quadruple p, q, r, s of points of M for which $pq + qr = pr$ is congruently (isometrically) embeddable in the euclidean plan E_2. In [3], Blumenthal showed that the euclidean weak four-point property characterizes inner-product spaces among the class of complete, convex, externally convex metric spaces. Blumenthal [4] introduced the euclidean feeble four-point property; namely each quadruple of points p, q, r, s of M for which $pq + qr = pr$ and $pq = qr$ is congruently embeddable in the euclidean plane E_2. He then showed that in a complete, convex, externally convex metric space the euclidean feeble four-point property implies the euclidean weak four-point property, thus obtaining another characterization of inner-product spaces. Day [6] showed the weaker condition, the euclidean queasy four-point property; for each pair of distinct points p, r of M there is a point q, $p \neq q \neq r$, between p and r such that for each point s of M the quadruple p, q, r, s is congruently embeddable in E_2, characterizes inner-product spaces among the class of complete, externally convex metric spaces. We will say a metric space has the local euclidean weak, feeble, queasy four-point property, respectively, provided for each point t of the space, there is a spherical neighborhood S_t with center t, such that

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S_t has the euclidean weak, feeble, or queasy four-point property. The main result of this paper is that a complete, convex, externally convex metric space is an inner-product space if and only if it has one of the aforementioned local four-point properties. It should be noted as in Day's work [6], that if the space has two distinct points, then the local euclidean queasy four-point property already assures the space is locally convex. We should also mention that Busemann [5, p. 51] asserts that every G-space which satisfies a local Pasch axiom has the local euclidean weak four-point property. Our main result shows this is false; for the two-dimensional Banach spaces with unique lines are all G-spaces which satisfy Pasch's axiom, but only one of them possesses the local euclidean weak four-point property.

Throughout this paper, M will denote a complete, convex, externally convex metric space. Also, for convenience, we will denote the local euclidean queasy four-point property, the local euclidean feeble four-point property, and the local euclidean weak four-point property, respectively by l.e.q.f.p.p., l.e.f.f.p.p. and l.e.w.f.p.p.

The following theorem, which is found in Busemann [5, p. 30] is useful in the sequel.

Theorem B. *If* $S(p; \rho/2)$ *is a sphere with center* p *and radius* $\rho/2$ *and if* x, y *are elements of* $S(p; \rho/2)$ *then any segment with endpoints* x *and* y *is contained in* $S(p; \rho)$.

2. **Some immediate consequences of l.e.q.f.p.p.** It is easily seen that if M has l.e.w.f.p.p. then M has l.e.f.f.p.p., which in turn implies M has l.e.q.f.p.p. We first show l.e.q.f.p.p. implies that distinct points a and b, with distance ab sufficiently small, lie on a unique metric segment. That l.e.q.f.p.p. implies l.e.w.f.p.p. then follows quite easily. Thus we show the three local properties are equivalent.

Theorem 2.1. *If* t *is an element of* M *and if* $S(t; \rho)$ *has the euclidean queasy four-point property, then each pair of distinct points in* $S(t; \rho/2)$ *are endpoints of exactly one metric segment.*

Proof. Since M is complete and convex, distinct points a, b in $S(t; \rho/2)$ are endpoints of at least one segment and by Theorem B, each such segment is contained in $S(t; \rho)$. Suppose a, b are endpoints of at least two segments; say $S_1(a, b)$ and $S_2(a, b)$. Let c be a point of $S_1(a, b) - S_2(a, b)$. In traversing $S_1(a, b)$ from c to a, a first point d of $S_2(a, b)$ is encountered, while a first point e of $S_2(a, b)$ is similarly met in traversing $S_1(a, b)$ from c to b. Thus the subsegments $S_1(d, e)$ and $S_2(d, e)$ of $S_1(a, b)$ and

$S_2(a, b)$, respectively, have only the endpoints d, e in common. The euclidean queasy four-point property implies the existence of a point g between d and e such that for each point s in $S(t; \rho)$, d, e, g, s, is congruent to a quadruple of points of E_2. Since $S_1(d, e)$ and $S_2(d, e)$ have only d, e in common, g is not on one of the segments; say $S_1(d, e)$. For the particular point f on $S_1(d, e)$ such that $df = dg$ and $ef = eg$, the quadruple d, e, f, g is congruent to a quadruple of E_2. This is impossible. Therefore each pair of points of $S(t; \rho/2)$ are endpoints of exactly one segment.

Theorem 2.2. *If t is an element of M and if $S(t; \rho)$ has the euclidean queasy four-point property, then $S(t; \rho/2)$ has the euclidean weak four-point property.*

Proof. Let p, r be points of $S(t;\rho/2)$ and let $Q(p, r)$ be the set of points x between p and r for which the quadruple p, r, x, s is congruently embeddable in E_2, for each point s in $S(t;\rho/2)$. It follows from the continuity of the metric that $Q(p, r)$ is closed and hence complete. Moreover, if x, y are points of $Q(p, r)$, then for each point s in $S(t; \rho/2)$ each of the quadruples p, r, x, s, and p, r, y, s is congruently embeddable in E_2. Applying the euclidean law of cosines to the angle with vertex p and sides $S(p, s)$ and $S(p, r)$, using the points p, r, s and then p, x, s we have

$$(ps^2 + pr^2 - rs^2)/2ps \cdot pr = (ps^2 + px^2 - sx^2)/2ps \cdot px.$$

Similarly for the quadruple p, r, y, s we have

$$(ps^2 + pr^2 - rs^2)/2ps \cdot pr = (ps^2 + py^2 - sy^2)/2ps \cdot py.$$

It now follows that p, s, x, y is congruently embeddable in E_2. In exactly the same manner it is seen that r, s, x, y is congruently embeddable in E_2. Applying the euclidean queasy four-point property to the points x, y, we obtain a point z such that for each s in $S(t; \rho/2)$, x, y, z, s is congruently embeddable in E_2. Use of the above techniques shows p, r, z, s is congruently embeddable in E_2. Thus $Q(p, r)$ is a complete, convex metric space and consequently contains a segment with endpoints p and r. By Theorem 2.1 there is only one such segment. It now follows that each quadruple of points of $S(t;\rho/2)$ which contains a linear triple is congruently embeddable in E_2; that is $S(t;\rho/2)$ has the euclidean weak four-point property.

We may now conclude that M has one of the local euclidean four-point properties if and only if it has all of them.

3. **Uniqueness of lines.** In the preceding section we proved l.e.q.f.p.p. implies unique segments, locally. It is now possible to extend this globally

and show that M has a unique metric line through each pair of its distinct points.

Theorem 3.1. *If M has l.e.w.f.p.p. and if a, b are distinct points of M, then a, b are endpoints of exactly one segment.*

Proof. If M contains distinct points a, b which are endpoints of two distinct segments, then as in the proof of Theorem 2.1, distinct points d, e and segments $S_1(d, e)$ and $S_2(d, e)$ can be found such that $S_1(d, e)$ and $S_2(d, e)$ have only the endpoints d, e in common. Since M is externally convex the segment $S_1(d, e)$ can be prolonged through e to a point f to obtain a segment $S_1(d, f)$ containing e and such that $df = 2de$. Since M has l.e.w.f.p.p. there is a sphere $S(e;\rho)$ with center e and radius ρ which has the euclidean weak four-point property. Since e is between d and f, e is between f and each point that is between d and e. Let p, q denote points on $S_1(d, e)$ and $S_2(d, e)$, respectively, such that $pe = qe = \rho/4$ and let r denote the point on $S_1(d, f)$ such that erf holds and $er = \rho/4$. Then the quadruple p, q, r, e is congruently embeddable in E_2 contrary to uniqueness of lines in E_2. Thus segments are unique.

Theorem 3.2. *If M has l.e.w.f.p.p., then metric segments admit unique prolongations.*

Proof. Since M is externally convex, segments admit prolongations. If M contained a segment with endpoints a, b which admitted two prolongations, then M would contain points a, b, c, d with b the midpoint of a and c and b the midpoint of a and d. Applying l.e.w.f.p.p., we obtain a sphere $S(b;\rho)$ which has the euclidean weak four-point property. We can now find points p, q, r, such that $pb = qb = rb = \rho/2$ and such that b is the midpoint of p and r and also of q and r. Now p, q, r, b is congruently embeddable in E_2 contradicting the uniqueness of lines in E_2.

Combining the results of Theorems 3.1 and 3.2 we have the following result.

Theorem 3.3. *If M has l.e.w.f.p.p., then each pair of distinct points of M lie on exactly one metric line.*

Since the local properties are equivalent, Theorem 3.3 remains valid if l.e.w.f.p.p. is replaced by l.e.f.f.p.p. or l.e.q.f.p.p.

4. **The characterization of M.** Since a local four-point property in a normed linear space is global, it suffices to show M is a Banach space. We show the Banach nature of M by showing l.e.w.f.p.p. implies that for

any three points p, q, r of M, if q', r' denote the respective midpoints of p and q and p and r, then $q'r' = qr/2$. This will then complete our characterization. For Andalafte and Blumenthal [1] called the latter property the Young postulate and proved a complete metric space with a unique line through each pair of its distinct points is a Banach space if and only if it satisfies the Young postulate. In § 3, we have already shown the uniqueness of lines.

Theorem 4.1. *Let t be an element of M and suppose $S(t; \rho)$ has the euclidean weak four-point property. If p, q, r are points in $S(t; \rho/2)$ and if q', r' are points on the lines joining p and q and p and r, respectively, such that $pq' = \lambda pq$ and $pr' = \lambda pr$, and if q', r' lie in $S(t; \rho/2)$, then $q'r' = \lambda qr$.*

Proof. Since p, q, q' are linear, the quadruple p, q', q, r is congruently embeddable in E_2. Thus using the euclidean law of cosines for the cosine of the angle with vertex p and sides $S(p, q)$ and $S(p, r)$, we obtain the same result whether using p, q, r or p, q', r. Similarly, the quadruple p, q', r', r is congruently embeddable in E_2, so the cosine law gives the same result for the angle at p when evaluating it with the points p, q', r or p, q', r'. Equality of the evaluation by the law of cosines in the first and last triangles yields $q'r' = \lambda qr$.

Corollary 4.2. *Suppose $S(t; \rho)$ has the euclidean weak four-point property and p, q, r are elements of $S(t; \rho/2)$. For each λ, $0 < \lambda < 1$ and for each point s between q and r if q', r' are between p and q and p and r, respectively, with $pq' = \lambda pq$, $pr' = \lambda pr$, then there is a point s' between q' and r' that is also between p and s and $ps' = \lambda ps$.*

Proof. Since p, q, r lie in $S(t; \rho/2)$, the segments joining each pair of points lie in $S(t; \rho)$, by Theorem B. Let s' be the point on $S(p, s)$ such that $ps' = \lambda ps$. Then by Theorem 4.1, $q's' = \lambda qs$, $s'r' = \lambda rs$, and $q'r' = \lambda qr$. The result now follows.

Theorem 4.3. *If M has l.e.w.f.p.p., p, x, y are points of M, and if x', y' are the midpoints of p and x and p and y, respectively, then $x'y' = xy/2$.*

Proof. Since M has l.e.w.f.p.p., a positive number ρ exists such that $S(p; \rho)$ has the euclidean weak four-point property. Let $\Lambda = \{\lambda > 0 \mid$ for each μ, $0 < \mu \leq \lambda$, for all s, t in $S(p; \rho/4)$ whenever s', t' are points on the lines joining p and s and p and t, respectively, such that (i) if $ps' = \mu \cdot ps$ and $pt' = \mu \cdot pt$, then $s't' = \mu \cdot st$, and (ii) for each point m between s and t, there is a point m' between s' and t' that satisfies the same betweeness relation as p, s, s' and $pu' = \mu \cdot pm$ and $m's' = \mu \cdot ms\}$. If Λ is not bounded above then the theorem follows. By way of contradiction, suppose Λ is bounded above. By Theorem 4.1 and Corollary 4.2, $\Lambda \neq \emptyset$. Thus sup Λ exists, say equals l. It follows from the continuity of the metric that l is an element of Λ. Let q, r be any points of $S(p; \rho/4)$ and let u, v be points on the rays of p and q and p and r, respectively, such that $pu = l \cdot pq$ and $pv = l \cdot pr$. For each point w on the segment $S(u, v)$, there is a sphere $S(w; \rho_w)$ which has the euclidean weak four-point property. Since $S(u, v)$ is compact and the collection of spheres $S(w; \rho_w/2)$, w on $S(u, v)$ covers $S(u, v)$, there is a finite subcollection which covers $S(u, v)$, say $S_1(w_1; \rho_1/2)$, $S_2(w_2; \rho_2/2)$, $\cdots$, $S_n(w_n; \rho_n/2)$. Without loss of generality suppose $u \neq w_i \neq v$ $(i = 1, 2, \cdots, n)$ and assume $w_i w_{i+1} w_{i+2}$ holds $(i = 1, 2, \cdots, n-2)$. Note uw_1w_2 and $w_{n-1}w_{n-2}v$ also hold. Let $s_0 = u$, $s_1 = w_1$, s_2 a point between w_1 and w_2 that is in both $S_1(w_1; \rho_1/2)$ and $S_2(w_2; \rho_2/2)$, $s_3 = w_2$, s_4 a point between w_2 and w_3 that is in both $S_2(w_2; \rho_2/2)$ and $S_3(w_3; \rho_3/2)$, and so forth. We thus obtain a finite collection of points $\{s_m\}$ of $S(u, v)$ with three points in the same sphere $S(w_k; \rho_k)$ and $s_i s_{i+1} s_{i+2}$ holds $i = 1, s, \cdots, m-2$. Consider the rays ps_i, $i = 0, 1, 2, \cdots, m$. It is easily seen that a positive number α exists such that the points s_i' and s_i'' on the ray of ps_i for which $s_i' s_i s_i''$ subsists and $ps_i' = \alpha \cdot ps_i$ and $ps_i = \alpha \cdot ps_i''$ lie in $S(w_j; \rho_j)$ whenever s_i does. Since s_i' is between p and s_i, it follows from Corollary 4.2 that the s_i' $(i = 0, 1, \cdots, m)$ are collinear. Moreover, the points p, s_i', s_{i+1}', s_i, s_{i+1} are congruently embeddable in E_2 since the mutual distances determined by them are just a constant multiple of the distances of corresponding points in $S(p; \rho/2)$. Easy applications of the euclidean weak four-point property using the linear triple s_i', s_i, s_i'' and one of s_{i+1}', s_{i+1}, s_{i+1}'' and the linear triple s_{i+1}', s_{i+1}, s_{i+1}' and one of s_i', s_i, s_i'' shows the above congruent embedding can be extended to see that p, s_i', s_{i+1}', s_i, s_{i+1}, s_i'', s_{i+1}'' are congruently embeddable in E_2. Since s_i, s_{i+1}, s_{i+2}, s_i', s_{i+1}', s_{i+2}', s_i'', s_{i+1}'', s_{i+2}'' lie in the same $S_j(w_k; \rho_j/2)$, it follows that $s_i'' s_{i+1}'' s_{i+2}''$ holds $(i = 0, 1, \cdots, m)$. Since $pu = \alpha \cdot ps_0''$, $\alpha < 1$, and $pu = l \cdot pq$, we have $ps_0'' = (l/\alpha)pq$ and $ps_m'' = (l/\alpha)pr$ and, from the above, $s_0'' s_m'' = (l/\alpha)qr$. It is clear from the above construction that if $l < \mu < l/\alpha$ and q', r' are points on the segments $S(p; s_0'')$ and $S(p; s_m'')$ with $pq' = \mu \cdot pq$, $pr' = \mu \cdot pr$, then $q'r' = \mu \cdot qr$. This contradicts $l = \sup \Lambda$. Therefore Λ is not bounded above; the proof is complete.

Applying the result of Andalafte and Blumenthal [1] we now have the following result.

Theorem 4.4. *If M has l.e.w.f.p.p., then M is a Banach space.*

The following theorem follows from Theorems 2.2 and 4.4.

Theorem 4.5. *A complete, convex, externally convex metric space is an inner-product space if and only if it has the local euclidean weak, feeble, or queasy four-point property.*

The authors [7] gave a local characterization of inner-product spaces among the class of normed linear spaces. So far as they are aware, the present paper gives the only local characterizations of inner-product spaces among a certain class of metric spaces. The proof given here makes strong use of the Young postulate. It would be nice to have a proof that is independent of that postulate, thus removing the necessity of first showing that local properties imply the metric space is a Banach space. If such a proof were obtained, it would probably be possible to obtain similar local characterizations of hyberbolic and spherical spaces. In [2] Andalafte and Valentine characterized euclidean and hyperbolic spaces among a certain class of metric spaces using intrinsic four-point properties. If such a strong use of the Young postulate could be avoided in the present work, it is conceivable that local intrinsic four-point properties would also effect a characterization of those spaces.

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