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FIRST CATEGORY FUNCTION SPACES UNDER THE TOPOLOGY OF POINTWISE CONVERGENCE

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ABSTRACT. A large class of function spaces, under the topology of pointwise convergence, are shown to be of first category.

The question as to when a space of continuous functions is of first category (i.e., can be written as a countable union of nowhere dense subsets) seems to be relatively unanswered. When the topology on the function space is the supremum metric topology, then if the range space is completely metrizable, so is the function space. Thus by the Baire category theorem, there is a large class of function spaces having the supremum metric topology which are Baire spaces (i.e., no open subspace is of first category). However, an example is given in [4] of a metrizable Baire space Y such that the space of continuous functions from I, the closed unit interval, into Y, under the supremum metric topology, is of first category. If the domain space is compact, the supremum metric topology agrees with the compact-open topology on the function space, so there is also a large class of function spaces having the compact-open topology which are Baire spaces. However, the situation changes dramatically when the topology of pointwise convergence is imposed on the function spaces. Under this topology, the function spaces are of first category for most nonpathological domain and range spaces. For example, it will follow from the Theorem in this paper that the space of real-valued continuous functions on I, with the topology of pointwise convergence, is of first category.

The notation $C_p(X, Y)$ will stand for the space of all continuous functions from X into Y under the topology of pointwise convergence. This topology is generated by the base

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{n} [x_i, V_i] \,\middle|\, x_i \in X \text{ and } V_i \text{ is open in } Y \right\},$$

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where each $[x, V] = \{f \in C_p(X, Y) \mid f(x) \in V\}$. It can also be considered as the topology inherited from the product topology, considering $C_p(X, Y)$ as a subspace of $\Pi_{x \in X} Y_x$, where each Y_x is a copy of Y.

Because of the nature of the topology on $C_p(X, Y)$, it is much easier to work with basic open sets rather than arbitrary open sets. For this reason we shall use a characterization of spaces of first category given in [5]. Basically, the proof of this characterization is due to Oxtoby [6]. Similar characterizations can be found for Baire spaces in [2] and [3].

If X is a topological space, a *pseudo-base* for X is a collection of nonempty open subsets of X such that each nonempty open subset of X contains a member of this collection.

Let $\mathcal{P}$ be a pseudo-base for X. Define $S[X, \mathcal{P}] = \{f: \mathcal{P} \to \mathcal{P} \mid f(U) \subset U$ for every $U \in \mathcal{P}\}$. If $U \in \mathcal{P}$ and $f, g \in S[X, \mathcal{P}]$, then define

$$\langle U, f, g \rangle_1 = g(U),$$

and for $i > 1$,

$$\langle U, f, g \rangle_i = \begin{cases} f(\langle U, f, g \rangle_{i-1}), & \text{if } i \text{ is even,} \\ g(\langle U, f, g \rangle_{i-1}), & \text{if } i \text{ is odd.} \end{cases}$$

Proposition. *The following are equivalent:*

(i) *X is of first category.*

(ii) *For every pseudo-base $\mathcal{P}$ for X, there exists an $f \in S[X, \mathcal{P}]$ such that for every $U \in \mathcal{P}$ and $g \in S[X, \mathcal{P}]$, $\bigcap_{i=1}^{\infty} \langle U, f, g \rangle_i = \varnothing$.*

(iii) *There exists a pseudo-base $\mathcal{P}$ for X and an $f \in S[X, \mathcal{P}]$ such that for every $U \in \mathcal{P}$ and $g \in S[X, \mathcal{P}]$, $\bigcap_{i=1}^{\infty} \langle U, f, g \rangle_i = \varnothing$.*

We shall call space X *completely Hausdorff with respect to* Y if for every finite set $\{x_1, \cdots, x_n\}$ of distinct points of X and every finite set $\{V_1, \cdots, V_n\}$ of nonempty open subsets of Y, there exists a continuous function $f: X \to Y$ such that $f(x_i) \in V_i$ for every $i = 1, \cdots, n$. Note that X is completely Hausdorff with respect to the reals if and only if it is completely Hausdorff in the usual sense.

Theorem. *Let X contain a convergent sequence which is infinite as a subset of X, and let X be completely Hausdorff with respect to Y. If Y satisfies any of the following three conditions, then the space $C_p(X, Y)$ is of first category.*

(i) *There exist two nonempty open subsets of Y with disjoint closures.*

(ii) *There exists a sequence* $\{W_i\}$ *of nonempty open subsets of* Y *such that, for each sequence* $\{y_i\}$ *in* Y *with* $y_i \in W_i$ *for every* i, *no subsequence of* $\{y_i\}$ *converges.*

(iii) Y *is of first category.*

Proof. Let $\{t_i\}$ be a convergent sequence in X which is infinite as a subset of X. To prove part (i), let U and V be open subsets of Y with disjoint closures. Let ω denote the set of positive integers, and let $\mathfrak{N} = \{N \subset \omega \mid \text{either } N \text{ is finite or } \omega \setminus N \text{ is finite}\}$. Now $\mathfrak{N}$ is countable, so let $\{N_n\}$ be an indexing of $\mathfrak{N}$ on ω. For each n, $i \in \omega$, let $W_n^i = U$ if $i \in N_n$ and let $W_n^i = V$ if $i \notin N_n$. For each $n \in \omega$, define $W_n = \bigcup_{i=1}^{\infty} [t_i, W_n^i]$, which is a dense open subset of $C_p(X, Y)$ since X is completely Hausdorff with respect to Y. Now suppose $f \in \bigcap_{n=1}^{\infty} W_n$. Let $M_1 = \{i \in \omega \mid f(t_i) \in U\}$, and let $M_2 = \{i \in \omega \mid f(t_i) \in V\}$. Suppose that M_1 is finite, so that $\omega \setminus M_1 = N_k$ for some $k \in \omega$. Now $f \in W_k$, so that for some $i \in \omega$, $f(t_i) \in W_k^i$. If $i \in M_1$, then $f(t_i) \in V$, so that $i \notin M_1$. But if $i \notin M_1$, then $f(t_i) \in U$, so that $i \in M_1$. Either way is a contradiction, so that M_1 cannot be finite. Similarly M_2 cannot be finite. But then $\{f(t_i)\}$ could not converge, which contradicts the continuity of f. Therefore $\bigcap_{n=1}^{\infty} W_n = \emptyset$, so that $C_p(X, Y)$ must be of first category in this case.

To prove part (ii), let $\{W_i\}$ be a sequence of nonempty open subsets of Y such that, for each sequence $\{y_i\}$ in Y with $y_i \in W_i$ for every $i \in \omega$, no subsequence of $\{y_i\}$ converges. Now if $\mathfrak{B}$ is the base for $C_p(X, Y)$ defined above, then each element of $\mathfrak{B}$ is nonempty since X is completely Hausdorff with respect to Y. Hence $\mathfrak{B}$ is a pseudo-base for $C_p(X, Y)$. Define $f \in S[C_p(X, Y), \mathfrak{B}]$ as follows. Let $V = \bigcap_{i=1}^{n} [x_i, V_i] \in \mathfrak{B}$ be arbitrary. Let $m(V)$ denote the smallest positive integer such that $t_{m(V)} \notin \{x_1, \cdots, x_n\}$. Now let $x_{n+1} = t_{m(V)}$ and $V_{n+1} = W_{m(V)}$. Then define $f(V) = \bigcap_{i=1}^{n+1} [x_i, V_i]$, which is clearly contained in V. With f thus defined, let $U \in \mathfrak{B}$ and $g \in S[C_p(X, Y), \mathfrak{B}]$. For each $i \in \omega$, let $x_i = t_{m(\langle U, f, g\rangle_{2i-1})}$. Since $\{x_i\}$ is a subsequence of $\{t_i\}$, it is convergent. But if ϕ were an element of $\bigcap_{i=1}^{\infty} \langle U, f, g\rangle$, then $\phi(x_i) \in W_{m(\langle U,f,g\rangle_{2i-1})}$ for every $i \in \omega$. Then by choice of $\{W_i\}$, $\{\phi(x_i)\}$ would not converge, which contradicts the continuity of ϕ. Thus $\bigcap_{i=1}^{\infty} \langle U, f, g\rangle_i = \emptyset$, so that $C_p(X, Y)$ must be of first category by the Proposition.

Part (iii) follows from considering $C_p(X, Y)$ to be a subspace of $\prod_{x \in X} Y_x$, which is dense since X is completely Hausdorff with respect to Y, and using the fact that a dense subspace of a space of first category is of first category.

Corollary 1. *Let X be a completely Hausdorff space which contains a convergent sequence which is infinite as a subset of X, and let Y be a nondegenerate pathwise connected Hausdorff space. Then $C_p(X, Y)$ is of first category.*

Corollary 2. *If X is a nondegenerate pathwise connected completely Hausdorff space, and Y is a nondegenerate locally pathwise connected Hausdorff space, then $C_p(X, Y)$ is of first category.*

The first corollary follows from the fact that a completely Hausdorff space is completely Hausdorff with respect to a pathwise connected space, and the second corollary follows from the first and the fact that the path-components of Y are open and from the Banach category theorem, which says that the union of any family of open subspaces of first category is of first category.

The three concepts given in statements (i), (ii), and (iii) of the Theorem overlap, but no two of them contain the third. Note that every sequentially compact space fails to satisfy (ii), and every quasiregular space which is not a Baire space does satisfy (ii). A closed interval is an example of a space satisfying (i) but not (ii) or (iii). For an example of a space satisfying (ii) but not (i) or (iii), take X to be the reals with topology generated by the usual open sets not containing o along with sets containing o and having countable complements. Finally, a countably infinite set with the cofinite topology satisfies (iii) but not (i) or (ii).

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