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EXTENSIONS OF ABELIAN GROUPS OF FINITE RANK

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ABSTRACT. Every abelian group X of finite rank arises as the middle group of an extension $e: 0 \to F \to X \to T \to 0$ where F is free of finite rank n and T is torsion with the p-ranks of T finite for all primes p. Given such a T and F we study the equivalence classes of such extensions which result from stipulating that two extensions $e_i: 0 \to F \to X_i \to T \to 0$, $i = 1, 2$, are equivalent if $e_1 = \beta e_2 \alpha$ for $\alpha \in \operatorname{Aut}(T)$ and $\beta \in \operatorname{Aut}(F)$. We reduce the problem to T p-primary of finite rank, where in the one case T is injective, and in the other case T is reduced. Suppose $T = \prod_{i=1}^m T_i$. In our main theorems we prove that in each case these equivalence classes of extensions are in 1-1 correspondence with the equivalence classes of n-generated subgroups of E where $E = \prod_{i=1}^m E_i$, $E_i = \operatorname{End}(T_i)$. Two n-generated subgroups of E will be called equivalent if one can be mapped onto the other by an automorphism of E.

1. With few exceptions our notation will be that of [1].

One of the difficulties in studying groups of extensions $\operatorname{Ext}(A, B)$ is that the same group may appear as a middle group in two distinct elements of $\operatorname{Ext}(A, B)$. With this in mind we consider the action of the rings $\operatorname{End}(A)$ and $\operatorname{End}(B)$ on $\operatorname{Ext}(A, B)$. In the problem at hand we are interested in the equivalence classes of $\operatorname{Ext}(T, F)$ induced by the action of automorphisms of T and F where T is a torsion group with its p-ranks finite for all primes p and F a free abelian group of rank n. Since $\operatorname{Ext}(T, F) = \prod_p \operatorname{Ext}(T_p, F)$, where T_p is the p-primary component of T, one may restrict himself to the case when T is p-primary. (See the remark following Theorem 1.2.) Since T is the direct sum of an injective group of finite rank and one which is finite we may treat each case separately. We now state our main theorems.

Theorem 1.1. *Let T be an injective p-primary group of rank m. Then the equivalence classes of extensions of the form $0 \longrightarrow F \longrightarrow X \longrightarrow T \longrightarrow 0$ are in 1-1 correspondence with the equivalence classes of n-generated subgroups of E.*

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In the other case we have

Theorem 1.2. *Let T be a reduced p-primary group of finite rank. Then the equivalence classes of extensions of the form $0 \to F \to X \to T \to 0$ are in* 1-1 *correspondence with the equivalence classes of n-generated subgroups of T.*

Remark. In the above theorems when T is not primary one must consider a sequence of such classes, one for each prime. Alternatively, Theorems 1.1 and 1.2 hold if one replaces "p-primary" in them by "torsion" and interprets rank in the sense of [2], [3]. When T is reduced and torsion an n-generated subgroup of T is just a subgroup of T of rank at most n (over Z) in the sense of [2], [3].

We begin with a proposition of interest, the proof of which is easy and will be omitted.

Proposition 1.3. (a) *Let M be a torsion group of rank one. Then* $\mathrm{Ext}(M, Z)$ *is a free* $\mathrm{End}(M)$*-module of rank one.*

(b) *Let t be an integer not exceeding s. Then* $\mathrm{Ext}(Z(p^t), Z(p^s))$ *is a free* $\mathrm{End}(Z(p^t))$*-module of rank one.*

Throughout the remainder of the paper we shall use the following notation. Let $T = \prod_{i=1}^m T_i$, $E_i = \mathrm{End}(T_i)$, $E = \prod_{i=1}^m E_i$, $V = Q \otimes F$, $S = \mathrm{End}(T)$ and $R = \mathrm{End}(F)$. We now give an outline of

Proof[1] of Theorem 1.1. The 1-1 correspondence of Theorem 1.1 is a composite of three 1-1 correspondences. The first of these makes correspond to the equivalence class of $e \in \mathrm{Ext}(T, F)$, an equivalence class of $n \times m$ matrices of p-adic integers in $\mathrm{Hom}(T, V/F)$. Here two $n \times m$ matrices are equivalent if one is obtained from the other by multiplying on the left by an $n \times n$ invertible matrix of integers and on the right by an $m \times m$ invertible matrix of p-adic integers. By taking transposes we can view the matrix in $\mathrm{Hom}(F, E)$. Our final correspondence sends the equivalence class of matrices in $\mathrm{Hom}(F, E)$ to the class of n-generated subgroups of E as in Theorem 1.1 by taking the class of the image of F in E.

We now prove these 1-1 correspondences. Our next proposition proves the first correspondence and in fact is a generalization of that result.

Proposition 1.4. $\mathrm{Ext}(T, F)$ *and* $\mathrm{Hom}(T, V/F)$ *are isomorphic $R - S$-bimodules.*

[1] The authors are grateful to the referee, whose careful reading of the paper resulted in an improvement in the proof of Theorem 1.1.

Proof. Apply the functor $\mathrm{Hom}(T, -)$ to the exact sequence $0 \to F \to V \to V/F \to 0$. This results in a group isomorphism θ: $\mathrm{Hom}(T, V/F) \cong \mathrm{Ext}(T, F)$, the map θ being the connecting homomorphism. Since θ is natural in T, i.e. $\theta(\sigma\delta) = \theta(\sigma)\delta$ for $\delta \in S$ and $\sigma \in \mathrm{Hom}(T, V/F)$, it is a map of right S-modules. It remains to show that it is also a map of left R-modules.

Since every endomorphism of a free abelian group of finite rank is a sum of automorphisms it suffices to show that $\theta(\rho\phi) = \rho\theta(\phi)$ holds for the units ρ in R and elements ϕ in $\mathrm{Hom}(T, V/F)$. Now $\theta(\phi)$ is represented by the sequence

$$\begin{tikzcd}
e\phi\colon 0 \arrow[r] & F \arrow[r] \arrow[d, equal] & (V, \phi) \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "\phi"] & 0 \\
e\colon 0 \arrow[r] & F \arrow[r] & V \arrow[r, "\pi"] & V/F \arrow[r] & 0
\end{tikzcd}$$

where $(V, \phi) = \{(x, t) | \pi(x) = \phi(t)\}$. Similarly $\theta(\rho\phi)$ can be represented by a sequence

$$0 \to F \to (V, \rho\phi) \to T \to 0 \tag{1}$$

where $(V, \rho\phi) = \{(x, t) | \pi(x) = \rho\phi(t)\}$. We remark here that we are abusing the notation by referring to several homomorphisms as ρ. Note that ρ: $F \to F$ extends uniquely to an automorphism from $V \to V$ which in turn induces a homomorphism from $V/F \to V/F$. We shall refer to these as well as the inclusion $F \to (V, \rho\phi)$ given by $y \to (\rho(y), 0)$ as ρ. Since the diagram

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\rho"] \arrow[d, equal] & (V, \rho\phi) \arrow[r] \arrow[d, "\rho^{-1}"] & T \arrow[r] \arrow[d, "\phi"] & 0 \\
e\colon 0 \arrow[r] & F \arrow[r] & V \arrow[r] & V/F \arrow[r] & 0
\end{tikzcd}$$

commutes, where $\rho^{-1}(x, t) = \rho^{-1}(x)$, we conclude that $e\phi = \theta(\phi)$ can be represented by (1). Using this and the sequence (1) we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\rho"] \arrow[d, "\rho"'] & (V, \rho\phi) \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F \arrow[r] & (V, \rho\phi) \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

Since the top row here is $\theta(\phi)$, the bottom row must be $\rho\theta(\phi)$. By definition however, the bottom row is $\theta(\rho\phi)$. Thus $\theta(\rho\phi) = \rho\theta(\phi)$ as required.

Remark. We note that the above proof uses only the facts that T is torsion and that every endomorphism of a free abelian group of finite rank is a sum of automorphisms.

Before we get into our second correspondence we recall the following facts. If $A = A_1 \oplus \cdots \oplus A_r$ and $B = B_1 \oplus \cdots \oplus B_s$ with injections α_j: $A_j \rightarrow A$ and projections $\beta_i \colon B \rightarrow B_i$, the map $\phi \rightarrow \beta_i \phi \alpha_j$ is an isomorphism of $\mathrm{Hom}(A, B)$ onto the group of matrices $M_{s\times r}(A, B)$ having as entries in the (i, j) position the elements of $\mathrm{Hom}(A_j, B_i)$. In particular if $\Lambda = \mathrm{End}(A)$ then Λ is isomorphic as a ring to $M_{r\times r}(A)$ and $\Sigma = \mathrm{End}(B)$ is isomorphic to $M_{s\times s}(B)$. Moreover the $\Sigma - \Lambda$-bimodule $\mathrm{Hom}(A, B)$ is isomorphic in the obvious sense to the $M_{s\times s} - M_{r\times r}$-bimodule $M_{s\times r}$. We now prove a generalization of our second correspondence.

Proposition 1.5. *The $R - S$-bimodule* $\mathrm{Hom}(T, V/F)$ *is isomorphic to the $S - R$-bimodule* $\mathrm{Hom}(F, E)$.

Proof. Interpreting the above paragraph in the case when T is an injective p-primary group of rank m, E becomes a free module of rank m over the p-adic integers J, and we obtain an isomorphism of the $R - S$-bimodule $\mathrm{Hom}(T, V/F)$ with the $Z_{n\times n} - J_{m\times m}$-bimodule $J_{n\times m}$. Here we note that we are considering V/F as a module over the endomorphism ring of F, not of V/F, but the correspondence described above remains valid. On the other hand, the $S - R$-bimodule $\mathrm{Hom}(F, E)$ is isomorphic to the $J_{m\times m} - Z_{n\times n}$-bimodule $J_{m\times n}$. Taking transposes one gets an isomorphism, in the obvious sense, of the $R - S$-bimodule $\mathrm{Hom}(T, V/F)$ with the $S - R$-bimodule $\mathrm{Hom}(F, E)$. This concludes the proof.

Now we come to our final correspondence. This arises from the observation that an n-generated subgroup of E is just a homomorphic image of F in E. We state this as

Proposition 1.6. *The elements ϕ, ϕ' in* $\mathrm{Hom}(F, E)$ *have equivalent images if and only if $\phi' = \alpha\phi\beta$ for some $\alpha \in \mathrm{Aut}(E)$ and $\beta \in \mathrm{Aut}(F)$.*

Proof. Note that if $\phi' = \alpha\phi\beta$ then the images of ϕ and ϕ' are equivalent since they are mapped onto each other by α. Moreover, this correspondence is clearly onto. In order to prove that it is 1-1 we have to show that for $\phi, \phi' \in \mathrm{Hom}(F, E)$ if $\phi(F) \cong \phi'(F)$ under an automorphism σ of E then there exist automorphisms β of F and α of E such that $\alpha\phi\beta = \phi'$. Since $\phi(F)$ and $\phi'(F)$ are finitely generated torsion-free groups they are free. Thus one can decompose F in two ways; $F \cong \phi(F) \oplus F_1 \cong \phi'(F) \oplus F_2$ where $F_1 \cong$ kernel ϕ and $F_2 \cong$ kernel ϕ'. Note that $F_1 \cong F_2$ since $\phi(F) \cong \phi'(F)$. It is easy to see now that one can induce an isomorphim β of F such that the diagram below commutes:

$$\begin{tikzcd} F \cong \phi(F) \oplus F_1 \arrow[r, "\phi"] & \phi(F) \arrow[d, "\sigma = \alpha"] \\ F \cong \phi'(F) \oplus F_2 \arrow[u, dashed, "\beta"] \arrow[r, "\phi'"] & \phi'(F) \end{tikzcd}$$

where α is set to be equal to σ. This completes the proof of Proposition 1.6 and Theorem 1.1.

The above proof almost carries verbatim to prove Theorem 1.2. Unfortunately the 1-1 relation of Proposition 1.6 breaks down. We circumvent this problem by replacing F by $G = F/p^sF$ where p^s is the minimal annihilator of T and utilizing the module isomorphism of the following

Theorem 1.7. $\operatorname{Ext}(T, F)$ *and* $\operatorname{Ext}(T, G)$ *are isomorphic as modules over the isomorphic rings* $\operatorname{End}(F) \otimes \operatorname{End}(T)$ *and* $\operatorname{End}(G) \otimes \operatorname{End}(T)$.

Proof. First we show the isomorphism of the rings. Define ϕ: $\operatorname{End}(F) \otimes \operatorname{End}(T) \to \operatorname{End}(G) \otimes \operatorname{End}(T)$ by $\phi(\beta \otimes \alpha) = \bar{\beta} \otimes \alpha$ where $\bar{\beta}$ is the homomorphism making the diagram

$$\begin{tikzcd} F \arrow[r, "\beta"] \arrow[d, "\pi"'] & F \arrow[d, "\pi"] \\ G \arrow[r, dashed, "\bar{\beta}"] & G \end{tikzcd}$$

commute. It is straightforward to show that ϕ is an isomorphism. Since $p^sT = 0$ observe that $F \xrightarrow{\pi} G$ induces an isomorphism of groups $\operatorname{Ext}(T, F) \xrightarrow{\pi_*} \operatorname{Ext}(T, G)$ where $e \to \pi e$. This isomorphism is a module isomorphism since $\pi_*(\beta e \alpha) = \pi\beta e\alpha = \bar{\beta}\pi e\alpha = \bar{\beta}\pi_*(e)\alpha$.

Proof of Theorem 1.2. By Theorem 1.7 it suffices to give the proof for G in place of F. The proof of Theorem 1.1 applies here except for the obvious changes and the 1-1 relation of Proposition 1.6. To do that we suppose that $\phi(F)$ is generated by $w_1, \cdots, w_k$ independent elements. Let $a_1, \cdots, a_k \in G$ be such that $\phi(a_i) = w_i$, $i = 1, \cdots, k$. It is not hard to show that $a_1, \cdots, a_k$ are independent elements each of p-height zero. Thus $\langle a_1, \cdots, a_k\rangle$ is a summand of G. Similarly the same holds for the preimage of the generators of $\phi'(F)$. The proof is now completed as in Theorem 1.1.

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