

RIGHT GROUP CONGRUENCES ON A SEMIGROUP

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ABSTRACT. This paper develops necessary and sufficient conditions on an algebraic semigroup S in order that it will have nontrivial right group homomorphic images. A central notion used is that of how the normal subsemigroups associated with the group images of S relate to the right group images of S . The results presented thus extend those of R. R. Stoll. Where right group congruences exist, the structure of S is determined and the right group congruences are characterized in terms of group congruences and right zero congruences on S . Sufficient conditions are then found for the existence of a minimum right group congruence on S , and isomorphic right group congruences and the minimum right group congruence on S are described. Lastly, an application is made to regular semigroups whose idempotents form a rectangular band.

1. Introduction. This paper develops necessary and sufficient conditions on an algebraic semigroup in order that it will have nontrivial right group homomorphic images. Relative to these, the structure of S is determined, and the right group congruence on S is found. Lastly, sufficient conditions are given for the existence of a minimum right group congruence. A central notion considered is that of how the normal subsemigroups associated with the group images of the semigroup relate to the right group images of the semigroup. The results presented thus extend those found by R. R. Stoll [5].

The basic definitions and terminology are those of Clifford and Preston [1]. Also, $S \setminus I$ will denote set difference, and the symbol \parallel will indicate the end of a proof.

Recall that a semigroup S is called a *right group* if it is right simple and left cancellative [1, p. 37]. Additional descriptions are contained in

Lemma 1 [1, pp. 38, 39]. *The following assertions concerning a semigroup S are equivalent:*

- (1) S is a right group.
- (2) S is right simple and contains an idempotent.

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- (3) S is a direct product $G \times E$ of a group G and a right zero semigroup E .
 (4) S is a union of isomorphic disjoint groups such that the set of identity elements of the groups is a right zero subsemigroup of S .
 (5) S is regular and left cancellative.

When a congruence ρ is such that S/ρ is the maximal homomorphic image of S of type C , as in [2, p. 275] and [2, Theorem 11.25 (A), p. 276], then ρ will be called the *minimum congruence* on S of type C and S/ρ will be called the *maximum homomorphic image* of S of type C . In other words, S/ρ is the maximum C -image if and only if ρ is of type C and $\rho \subseteq \sigma$ for each congruence σ which is of type C . Moreover, the phrases "right group congruence", "group congruence", and "right zero congruence" will be denoted by RGC, GC and RZC respectively. When such a congruence is minimum, it will be denoted by MRGC, MGC, and MRZC respectively.

2. Right group congruences. Theorem 11.25 (A) of [2] would seem to indicate that the minimum right group congruence on a semigroup could be found by considering the intersection of all the right group congruences on S . However, this is not necessarily true as the following example shows.

Let S be the additive semigroup of positive integers. The group images of S are exactly all the finite cyclic groups, and there is no maximal group among these. The intersection of the induced congruences on S is the identity congruence ι , but ι is not a right group congruence as S is not right simple.

If σ is a GC on S , then the subset N of S , where N denotes the group identity of S/σ , is a subsemigroup of S and, moreover, N can be used in the following manner to generate σ .

Define the relation σ_N on S by

$$(2.1) \quad \sigma_N = \{(a, b) \in S \times S; ax, bx \in N \text{ for some } x \text{ in } S\}.$$

It can be verified that $\sigma = \sigma_N$. The subsets N which generate group images of S , relative to the relation defined in equation (2.1), R. Stoll [5, p. 478] calls, after Dubreil, the normal and unitary subsemigroups of S . They are also characterized by [5, Theorem 3, p. 477]:

- (i) N is a subsemigroup of S ;
 (ii) if $a, b \in S$ and $ab \in N$ then $ba \in N$;
 (iii) if $a, b \in S$ and a and $ab \in N$ then $b \in N$;
 (iv) N is a neat subset of S .

Recall from [2, p. 16] that a subset N is called *neat* if it is a right neat sub-

set of S and a left neat subset of S , i.e., if for each $s \in S$, there exists an $x \in S$ and $y \in S$ such that $sx \in N$ and $ys \in N$.

Clifford and Preston [2, pp. 55, 56] describe a subset N of S as being *reflexive* if it satisfies (ii) of (2.2) and *left unitary* if it satisfies (iii) of (2.2). Henceforth, a subsemigroup satisfying (i)–(iii) of S will be called a normal subsemigroup of S .

The following theorem relates the group homomorphisms of S and its neat normal subsemigroups.

Theorem 2.3 [5, Theorem 2, p. 476]. *The correspondence $S/\alpha \rightarrow (1_{S/\alpha})(\alpha^h)^{-1}$, where $1_{S/\alpha}$ is the unit of the group S/α , is a one-to-one correspondence between the set of (isomorphically) distinct group images and the set of neat normal subsemigroups of S .*

Theorem 2.4 [2, Theorem 10.24, p. 190]. *The group congruence determined by a neat normal subsemigroup N of S is precisely the σ_N of equation (2.1). In particular, N is the identity of S/σ_N .*

In trying to generalize the preceding work to right group images, it would seem natural to replace condition (iv) of (2.2) by the property that N be only a right neat subset of S . But since N would be reflexive by condition (ii) of (2.2), then N would also be left neat and hence a neat subset of S .

The following theorem develops the sought for characterizations of a RGC on a semigroup S , and gives some results on the inherent structure of S . A RGC will now be classified as being trivial if it is a group congruence or if it is a right zero congruence.

Theorem 2.5. *The following conditions on a semigroup S are equivalent.*

- (A) *There exists a nontrivial right group congruence on S .*
- (B) *S is a disjoint union of two or more left ideals, and contains a proper neat normal subsemigroup.*
- (C) *There exists a nontrivial group congruence ρ on S and a nontrivial right zero congruence π on S .*

Proof. (A) *implies* (B). Assume ρ is a nontrivial RGC on S and that $S/\rho \cong G \times E$ where G is a group and E is a right zero semigroup. Note that $E \cong E_{S/\rho}$. For each e in E , define L_e to be the preimage of $G \times \{e\}$, and define K to be the kernel of ρ , i.e., the union of all the ρ -classes meeting E . Evidently $S = \bigcup \{L_e : e \in E\}$, with the union disjoint. If $s \in L_e$, $t \in L_f$ and e, f in E , then

$$(st)\rho^h = (s\rho^h)(t\rho^h) \in (G \times \{e\})(G \times \{f\}).$$

But $(G \times \{e\})(G \times \{f\}) = G \times \{f\}$ and therefore $st \in L_f$, i.e., L_f is a left ideal of S . It can be shown that K satisfies (2.2).

(B) *implies* (C). Given a neat normal subsemigroup K and $\{L_e: e \in E\}$, a set of disjoint left ideals of S such that $S = \bigcup \{L_e: e \in E\}$, set $\alpha = \sigma_K$ as defined by equation (2.1), and let π be the RZC induced on S by $\{L_e: e \in E\}'$.

(C) *implies* (A). Define the congruence ρ on S by $\rho = \alpha \cap \beta$, where α is the GC and β is the RZC on S . For a, b in S there exists a group inverse $x\alpha$ of $a\alpha$. Since $ax\alpha$ is the identity of S/α , then $axb\alpha b$. But $axb\beta b$ and thus $axb\rho b$, i.e., ρ is a right simple congruence on S . It follows from Lemma 1(2) that ρ is a RGC on S . \parallel

Thus, a RGC ρ induces a GC σ_K , where K is the kernel of ρ , and a RZC π , where the π -classes are left ideals of S , and conversely. The exact relationship between ρ , σ_K , and π is exhibited in

Theorem 2.6. *Let ρ be a right group congruence on a semigroup S , and let $\alpha: S/\rho \rightarrow G \times E$ be an isomorphism where G is a group and E is a right zero semigroup. Let $K = \text{kernel } \rho^h \circ \alpha$, and for each $e \in E$, let L_e be the preimage of $G \times \{e\}$, and denote by π the right zero congruence induced on S by the L_e 's. Then:*

$$(A) \rho = \sigma_K \cap \pi.$$

$$(B) S/\rho \cong S/\sigma_K \times S/\pi.$$

(C) *For each e in E , $K \cap L_e$ is a neat normal subsemigroup of L_e .*

$$(D) S/\sigma_K \cong L_e/\sigma_{K \cap L_e} \text{ for each } e \text{ in } E.$$

Proof. Following the notation of Theorem 2.5, K is the preimage of $\{1_G\} \times E$, and for each e in E , the preimage of $G \times \{e\}$ is a left ideal of S .

(A). If apb , then $ap^h = (g, e) = bp^h$ for some (g, e) in $G \times E$.

There exists an x in S such that $x\rho^h = (g^{-1}, e)$, and therefore $xap^h = (g^{-1}, e) \cdot (g, e) = (1_G, e)$ and similarly $xbp^h = (1_G, e)$, i.e., $xa, xb \in K$ and therefore $a\sigma_K b$. Since $ap^h, bp^h \in G \times \{e\}$, $a\pi b$ and thus $\rho \subseteq \sigma_K \cap \pi$.

Conversely, $(a, b) \in \sigma_K \cap \pi$ implies that there exists e in E such that $ap^h, bp^h \in G \times \{e\}$, and that there exists x in S such that $xa, xb \in K$. Thus $xap^h = (1_G, e) = xbp^h$. If $x \in L_f$, say, then there exists $x' \in S$ such that $x'\rho^h = (1_G, f)$, i.e., $x'x\rho \in E_{S/\rho}$. Multiplying by $x'\rho^h$ yields:

$$a\rho^h = (1_G, f) \cdot ap^h = x'xap^h = x'xbp^h = (1_G, f) \cdot bp^h = bp^h,$$

i.e., apb , and $\sigma_K \cap \pi \subseteq \rho$.

(B). From part (A), $\rho = \sigma_K \cap \pi$ so define a map θ from S/ρ to $S/\sigma_K \times S/\pi$ by $(s\rho)\theta = (s\sigma_K, s\pi)$. Clearly, θ is well defined. For s, t in S ,

$(sp \cdot tp)\theta = (stp)\theta = (st\sigma_K, st\pi) = (sp)\theta \cdot (tp)\theta$ and hence θ is a homomorphism. For $(s\sigma_K, t\pi) \in S/\sigma_K \times S/\pi$, let $t \in L_e$ and $k \in K \cap L_e$. Thus $(skp)\theta = (s\sigma_K, t\pi)$, i.e., θ is onto. Lastly, if $(sp)\theta = (tp)\theta$ then $s\sigma_K t$ and $s\pi t$, and therefore spt , i.e., θ is one-to-one on S/ρ .

(C). Denote $K \cap L_e$ by K_e for each e in E , and consider the four conditions of (2.2). Conditions (i)–(iii) are evident so consider (iv). For a in L_e and k in K_e there exists an x in S such that $xka \in K$. But xk , $xka \in L_e$ since $a, k \in L_e$, and thus $xka \in K \cap L_e = K_e$. Thus K_e is a left neat subset of L_e and so by condition (ii) is also a right neat subset of L_e .

(D). For convenience, denote σ_{K_e} by σ_e . Let $e \in E$ and define the map θ from L_e/σ_e to S/σ_K by $(t\sigma_e)\theta = t\sigma_K$ for each t in L_e . Evidently, θ is a homomorphism. If $s \in S$ and $k \in K_e$, then $sk \in L_e$ and $(sk\sigma_e)\theta = sk\sigma_K = s\sigma_K$, i.e., θ is onto. Lastly, let $a, b \in L_e$ and suppose that $(a\sigma_e)\theta = (b\sigma_e)\theta$, i.e., that $a\sigma_K b$. There thus exists an x in S such that $ax, bx \in K$. For k in K_e , $axk, bxx \in K_e$ since $KK_e = K(K \cap L_e) \subseteq K \cap L_e = K_e$. Since $xk \in L_e$, then $a\sigma_e b$, i.e., θ is one-to-one. \parallel

3. The minimum right group congruence. Sufficient conditions for the existence of a minimum right group congruence are contained in the next theorem. For this, assume S is a semigroup having nontrivial right group homomorphs. By Theorem 2.5, S has neat normal subsemigroups and there exist decompositions of S into the disjoint union of left ideals of S .

Theorem 3.1. *Let S be a semigroup having nontrivial right group homomorphs. Denote the set of neat normal subsemigroups of S by $\{K_i; i \in I\}$, and the set of left ideal partitions of S by $\{P_j; j \in J\}$. Then:*

(A) *Two right group homomorphs are isomorphic if and only if their associated neat normal subsemigroups are equal and their associated left ideal decompositions of S have the same number of left ideals.*

(B) *If S has a minimum neat normal subsemigroup K , in the sense that K is contained in every other neat normal subsemigroup, then S has a (unique) maximum right group homomorph.*

Proof. For i in I let σ_i denote the RGC σ_{K_i} , and for j in J , let π_j denote the RZC induced on S by P_j . Define the RGC ρ_{ij} by $\rho_{ij} = \sigma_i \cap \pi_j$.

(A). Consider the pairs (K_h, P_j) and (K_i, P_k) where $h, i \in I$ and $j, k \in J$. If $K_h = K_i$ and $|P_j| = |P_k|$, then $\sigma_h = \sigma_i$ and $S/\pi_j \cong S/\pi_k$. Hence by Theorem 2.6(B)

$$S/\rho_{ij} \cong S/\sigma_i \times S/\pi_j \cong S/\sigma_i \times S/\pi_k \cong S/\rho_{ik}.$$

Conversely, for ρ_1 and ρ_2 such that $S/\rho_1 \cong S/\rho_2$, let $S/\rho_1 \cong G_1 \times E_1$ and $S/\rho_2 \cong G_2 \times E_2$. Since $E_1 \cong E_2$, then $|P_1| = |P_2|$ for the associated left ideal partitions P_1 and P_2 of S . Since $G_1 \cong G_2$, then by Theorem 2.3, $K_1 = K_2$.

(B) For each $j \in J$, the left ideal partition P_j induces a RZC, say π_j . Define the congruence π on S by $\pi = \bigcap \{\pi_j; j \in J\}$. If $a, b \in S$, then $ab\pi_j = b\pi_j$ for each $j \in J$, implies that $ab\pi = b\pi$, i.e., π is the MRZC on S . The MRGC on S is therefore $\sigma_k \cap \pi$. \parallel

The next result is an application [example] of Theorem 3.1. Let S be a regular semigroup and denote its set of idempotents by E . It is shown directly in [4, Theorem 2.7, p. 394] that E is a rectangular band if and only if the MRGC on S is given by $\rho = \{(a, b) \in S \times S; ea = eb \text{ for all } e \in E\}$. However, by Theorem 3.1, ρ should equal $\sigma \cap \pi$, where σ is the MGC on S and π is the MRZC on S . Corollary 3.3 proves this in a direct manner, but first a lemma. If $x \in S$, x' will denote an inverse of x .

Lemma 3.2. *If S is a regular semigroup whose set of idempotents, E , is a rectangular band, then:*

- (i) $xy = xy \forall x, y \in S$ and $\forall e \in E$.
- (ii) Sa is a minimal left ideal $\forall a \in S$.
- (iii) S is the disjoint union of its minimal left ideals.
- (iv) $a\pi = Sa \forall a \in S$, where π is the MRZC on S .

Proof. For (i), $xy = xx'xeyy'y$ since S is regular. Since E is a rectangular band, $x'xeyy' = x'xyy'$. Thus $xy = xx'xyy'y = xy$.

Next suppose that L is a left ideal of S contained in some Sa . For $x \in L$, there exists $s \in S$ such that $x = sa$. By (i), $a = aa'a = aa's'sa$, and it follows that $a = aa's'x$, i.e., $a \in Sx$. Thus $Sa \subseteq L$ and $L = Sa$, establishing (ii).

Since $a \in Sa$ for each $a \in S$, and since minimal left ideals are disjoint, then (iii) follows from (ii).

To prove (iv) assume that $a\pi$ contains Sa as a proper subset and denote $a\pi \setminus Sa$ by D . If $d \in D$ and $s \in S$, then $sd \in a\pi$. But $sd \notin Sa$, otherwise by (ii), $Sd = Sa$ and therefore $d \in Sa$, a contradiction. Thus D is a left ideal of S contained in $a\pi$. But this implies that the left ideal partition induced by π is not the maximum one, which contradicts the fact that π is the MRZC on S . Hence D is empty and $a\pi = Sa$. \parallel

Corollary 3.3. *Let S be a regular semigroup and denote the set of idempotents of S by E . If E is a rectangular band, then the following are equivalent.*

- (1) ρ is the minimum right group congruence on S .
 (2) $\rho = \{(a, b) \in S \times S : ea = eb \text{ for all } e \in E\}$.
 (3) $\rho = \sigma \cap \pi$, where σ is the minimum group congruence on S and π is the minimum right zero congruence on S .

Proof. By [4, Theorem 2.7, p. 394], (1) and (2) are equivalent. Recall from [4, Theorem 3.1, p. 396] that since E is a subsemigroup of S , the MGC on S is given by $\sigma = \{(a, b) \in S \times S : eae = ebe \text{ for some } e \in E\}$.

(2) \rightarrow (3). If apb then for all $e \in E$, $ea = eb$. Thus $eae = ebe$ and $(a, b) \in \sigma$. For $s \in S$, Lemma 3.2(i) implies that $sa = sea = seb = sb$, i.e., $Sa = Sb$. Hence by (iv) of Lemma 3.2, $(a, b) \in \pi$ and therefore $\rho \subseteq \sigma \cap \pi$.

Conversely, if $(a, b) \in \sigma \cap \pi$, there exists $e \in E$ such that $eae = ebe$, and $a\pi = b\pi$. By (iv) of Lemma 3.2 then, there exists $y \in S$ such that $b = ya$. Now note that $(ab'y)(ab'y) = ab'(ya)b'y = ab'(b)b'y = ab'y$, i.e., $ab'y \in E$. Since $eae = ebe$, then by Lemma 3.2(i), $eaeb'b = ebeb'b$ implies that $eab'b = eb$. Since $b = ya$ and $ab'y \in E$, it follows again from (i) of Lemma 3.2 that

$$eb = eab'b = ea(b'y a) = e(ab'y)a = ea.$$

Thus for any $f \in E$, $fa = fea = feb = fb$, i.e., $(a, b) \in \rho$.

That (3) \rightarrow (1) follows from Theorem 2.6(A). \parallel

Remark 3.4. Let $S = \{a_{11}, a_{21}\}$ and $a_{ij}a_{kl} = a_{il}$ for $i, j, k, l \in \{1, 2\}$. Then S is a regular semigroup whose idempotents form a rectangular band, but S has no nontrivial right group images since $\pi = \sigma = \omega$, i.e., $\rho = \omega$ in Theorem 6.1 and Corollary 3.3.

Remark 3.5. As an alternate approach to Corollary 3.3, it can be shown directly that E is a neat reflexive subsemigroup of S . Moreover, if $e, ea \in E$, then $a = aa'a = aa'(e)a$ by Lemma 3.2(i), and therefore $a = (aa')(ea) \in E$. Thus E is left unitary and hence a normal subsemigroup of S . Applying (2.2), the MGC of Corollary 3.3 is precisely σ_E , and it can be verified that $\sigma = \sigma_E$.

Some of the results presented above are an extension of results contained in the author's doctoral dissertation directed by Professor D. W. Miller of the University of Nebraska. The author also wishes to thank the referee for his valuable suggestions and remarks.

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