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A NONEMBEDDING THEOREM FOR ALGEBRAS

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ABSTRACT. A certain nonembedding result has previously been shown for Lie and associative algebras. This note gives a generalization and several other consequences are noted.

Recently, D. A. Towers has considered the Frattini subalgebra of a nonassociative algebra. The concept had been previously considered for Lie and associative algebras. The purpose of the present note is to give a generalization of the nonembedding results which appear in [7] and [8] and to observe some special cases.

Let A be a nonassociative algebra over a field F. For $x \in A$, denote by L_x the left multiplication of A by x. For any left ideal C of A, let $E(C, A) = \{L_x$ restricted to C, for all $x \in A\}$. For $\alpha, \beta \in F$, define a new product in A by letting $a \circ b = \alpha(ab) + \beta(ba)$ for all $a, b \in A$ and denote this nonassociative algebra by $A(\alpha, \beta)$. For $\alpha, \beta, \tau, \sigma \in F$, let

$$N(\alpha, \beta, \tau, \sigma) = \{x \in A;\ (L_{a\circ b} - \tau L_a L_b - \sigma L_b L_a)x = 0 \text{ for all } a, b \in A\}.$$

If $C \subseteq N(\alpha, \beta, \tau, \sigma)$, then in $E(C, A)$ define

$$L_a \circ L_b = \tau L_a L_b + \sigma L_b L_a = L_{a\circ b} \quad \text{for all } a, b \in A.$$

This is a well-defined product in $E(C, A)$ and the mapping $L : a \longrightarrow L_a$ is a homomorphism from $A(\alpha, \beta)$ into $E(C, A)$. Henceforth we consider $E(C, A)$ as an algebra under this product. Now let B be an ideal in A and define $r_1 = \{x \in B;\ L_y(x) = 0$ for all $y \in B\}$ and define r_i inductively by $r_i = \{x \in B;\ L_y(x) \in r_{i-1}$ for all $y \in B\}$. If $\tau \neq 0$ and $r_i \subseteq N(\alpha, \beta, \tau, \sigma)$, then r_i is a left ideal of A since if $z \in r_i$, $x \in A$, $y \in B$, then

$$L_y(xz) = \tau^{-1}(L_{x\circ y} - \sigma L_x L_y)(z) \in r_{i-1}.$$

Finally for any algebra D, let $\Phi(D)$ be the intersection of all maximal subalgebras of D if maximal subalgebras of D exist and let $\Phi(D) = D$ otherwise. If π is a homomorphism from D, then $\pi(\Phi(D)) \subseteq \Phi(\pi(D))$.

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Theorem. *Let B be a nonassociative algebra over a field F such that $\dim r_1 = 1$ and $1 < \dim r_2 < \infty$. Let $\alpha, \beta, \tau, \sigma \in F$ such that $\tau \neq 0$. Then there does not exist an algebra A such that B is an ideal of A, $B \subseteq \Phi(A(\alpha, \beta))$ and $r_2 \subseteq N(\alpha, \beta, \tau, \sigma)$.*

Proof. Suppose to the contrary that B is an ideal of A, $B \subseteq \Phi(A(\alpha, \beta))$ and $r_2 \subseteq N(\alpha, \beta, \tau, \sigma)$. Let L be the mapping of A onto $E(r_2, A)$ given by $L : a \to L_a$ restricted to r_2. Then L is a homomorphism of $A(\alpha, \beta)$ onto $E(r_2, A)$ and

$$E(r_2, B) = L(B) \subseteq L(\Phi(A(\alpha, \beta))) \subseteq \Phi(L(A(\alpha, \beta))) = \Phi(E(r_2, A)).$$

Let $z_1, \cdots, z_k$ be a basis for r_2 such that z_k is a basis for r_1. For $i = 1, \cdots, k-1$, let e_i be the linear transformation defined by

$$e_i(z_j) = \begin{cases} \delta_{ij} z_k & \text{for } j = 1, \cdots, k-1, \\ 0 & \text{for } j = k, \end{cases}$$

where δ_{ij} is the Kronecker delta, and let S be the vector space generated by $e_1, \cdots, e_{k-1}$. We claim that $S = E(r_2, B)$. Since $Br_2 \subseteq r_1$ and $Br_1 = 0$, $E(r_2, B) \subseteq S$. For each $x \in B$, L_x induces a linear transformation from r_2 into $r_1 \simeq F$. Therefore we consider each L_x, $x \in B$, as a linear functional on r_2. That is, $E(r_2, B) \subseteq (r_2)^*$ where r_2^* is the dual space of r_2. Consequently, $\dim E(r_2, B) = \dim r_2 - \dim r_2^B$ where $r_2^B = \{z \in r_2, L_x(z) = 0 \text{ for all } x \in B\}$. Clearly $r_2^B = r_1$. Hence $\dim E(r_2, B) = k - 1$ and $E(r_2, B) = S$.

We now show that S is complemented in $E(r_2, A)$ which contradicts $S \subseteq \Phi(E(r_2, A))$. Let

$$M = \left\{ E \in E(r_2, A);\ E(z_i) = \sum_{j=1}^{k-1} \lambda_{ij} z_j,\ \lambda_{ij} \in F,\ i = 1, \cdots, k-1 \right.$$
$$\left. \text{and } E(z_k) = \lambda_k z_k,\ \lambda_k \in F \right\}.$$

M is a subalgebra of $E(r_2, A)$ since if $L_x, L_y \in M$, then $L_x \circ L_y = \tau L_x L_y + \sigma L_y L_x \in M$. Also $M \cap S = 0$. To see that $M + S = E(r_2, A)$, let $E \in E(r_2, A)$. Then

$$E(z_i) = \sum_{j=1}^{k-1} \lambda_{ij} z_j + \lambda_{ik} z_k \quad \text{and} \quad E(z_k) = \lambda_k z_k.$$

However

$$E = E - \sum_{i=1}^{k-1} \lambda_{ik} e_i + \sum_{i=1}^{k-1} \lambda_{ik} e_i$$

where $E - \sum_{i=1}^{k-1} \lambda_{ik} e_i \in M$ and $\sum_{i=1}^{k-1} \lambda_{ik} e_i \in S$. Hence $M + S = E(r_2, A)$ which is the desired contradiction.

Henceforth, B is considered to be as in the Theorem. The right nucleus, N_r, of a nonassociative algebra A is the set of all $z \in A$ such that $L_{xy}(z) = L_x L_y(z)$ for all $x, y \in A$. That is, $N_r = N(1, 0, 1, 0)$.

Corollary 1. *B cannot be an ideal of any nonassociative algebra A such that $B \subseteq \Phi(A)$ and $r_2 \subseteq N_r$. In particular, B cannot be an ideal contained in $\Phi(A)$ for any associative algebra A.*

In a standard algebra A, the set C of all commutators is contained in the right nucleus [2].

Corollary 2. *B cannot be an ideal of any standard algebra A such that $B \subseteq \Phi(A)$ and $r_2 \subseteq C$.*

The J-nucleus, J, of any Malcev algebra A is the collection of all $x \in A$ such that $(L_z L_y + L_{yz} - L_y L_z)(x) = 0$ for all $y, z \in A$ [5]. Then $J = N(1, 0, 1, -1)$.

Corollary 3. *B cannot be an ideal of any Malcev algebra A such that $B \subseteq \Phi(A)$ and $r_2 \subseteq J$. In particular, B cannot be contained in $\Phi(A)$ for any Lie algebra A.*

For any Malcev algebra A, an associated Lie triple system is defined by

$$[x, y, z] = (2L_z L_y - L_{yz} + L_y L_z)(x) \quad \text{for all } x, y, z \in A.$$

The center of any Lie triple system T is the set of all $x \in T$ such that $[x, y, z] = 0$ for all $y, z \in T$. (See [3].) Then the center Z of the associated Lie triple system of a Malcev algebra is $N(-1, 0, -1, -2)$.

Corollary 4. *B cannot be an ideal of any Malcev algebra A such that $B \subseteq \Phi(A)$ and $r_2 \subseteq Z$.*

If A is left alternative of characteristic not 2, then the associated Jordan algebra A^+ is defined by $x \circ y = \frac{1}{2}(xy + yx)$. (See [1].) In the present notation $A^+ = A(\frac{1}{2}, \frac{1}{2})$. Then the following identity holds in A:

$$L_{x \circ y} = \tfrac{1}{2}(L_x L_y + L_y L_x).$$

Hence $A = N(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Corollary 5. *B cannot be an ideal of any left alternative algebra A such that $B \subseteq \Phi(A^+)$.*

The following examples satisfy the conditions on B in the Theorem.

Example 1. Let A be the collection of all strictly upper triangular n by n matrices with elements from a field. Let e_{ij} be the usual matrix unit. Let $B = A^+$, the associated Jordan algebra of A. Then $r_1 = ((e_{1n}))$ and $r_2 = ((e_{1n}, e_{1,n-1}, e_{2,n}))$. If $n \geq 4$, then B is not associative. Also if $B = A^-$, the associated Lie algebra, then $r_1 = ((e_{1n}))$ and $r_2 = ((e_{1,n}, e_{1,n-1}, e_{2,n}))$.

Example 2. Let B have basis $e, u, v, z_1, \cdots, z_n$. Define multiplication by $e^2 = e$, $ue = v$, $eu = u$, $(z_i)^2 = v$ and all other products between basis elements to be 0. Then B is left alternative as is seen by a straightforward computation. However B is not right alternative since $ue^2 \neq (ue)e$. Here $r_1 = ((v))$ and $r_2 = ((v, z_1, \cdots, z_n))$.

Example 3. This example is essentially taken from [5, p. 435]. Let B have basis $e_3, e_{i1}, e_{i2}, e_{i4}, e_{i5}$ for $i = 1, \cdots, n$. Define multiplication by

$$e_{i1}e_{i4} = -e_{i4}e_{i1} = e_{i2}, \qquad e_{i2}e_{i5} = -e_{i5}e_{i2} = e_3 \quad \text{for } i = 1, \cdots, n$$

and all other products between basis elements to be 0. Note that B is anticommutative and all products involving four elements are 0. Then B is Malcev (see [5, Proposition 2.21]). B is not Lie since

$$(e_{i1}e_{i4})e_{i5} + (e_{i4}e_{i5})e_{i1} + (e_{i5}e_{i1})e_{i4} = e_3.$$

Then $r_1 = ((e_3))$ and $r_2 = ((e_3, e_{i2}, e_{i5} \text{ for } i = 1, \cdots, n))$.

REFERENCES

1. A. A. Albert, *On right alternative algebras*, Ann. of Math. (2) 50 (1949), 318–328. MR 10, 503.
2. E. Kleinfeld, *Standard and accessible rings*, Canad. J. Math. 8 (1956), 335–340. MR 17, 1180.
3. W. G. Lister, *A structure theory of Lie triple systems*, Trans. Amer. Math. Soc. 72 (1952), 217–242. MR 13, 619.
4. O. Loos, *Über eine Beziehung zwischen Malcev-Algebren und Lie-Tripelsystemen*, Pacific J. Math. 18 (1966), 553–562. MR 33 #7385.
5. A. A. Sagle, *Malcev algebras*, Trans. Amer. Math. Soc. 101 (1961), 426–458. MR 26 #1343.
6. R. D. Schafer, *An introduction to nonassociative algebras*, Pure and Appl. Math., vol. 22, Academic Press, New York, 1966. MR 35 #1643.
7. E. L. Stitzinger, *A nonimbedding theorem of associative algebras*, Pacific J. Math. 30 (1969), 529–531. MR 40 #7307.
8. ———, *On the Frattini subalgebra of a Lie algebra*, J. London Math. Soc. (2) (1970), 429–438. MR 41 #8484.
9. D. A. Towers, *A Frattini theory for algebras*, Proc. London Math. Soc. (3) 27 (1973), 440–462.

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