PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, July 1975

THE MODEL COMPANION OF ZF

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ABSTRACT. We prove that the theory ZF has a model companion and we describe an axiom system for it.

The notion of a model companion was introduced by E. Bers as a generalization of the notion of a model completion [3, §5]. In this paper we prove that ZF has a model companion and describe a set of axioms for it. This model companion, however, resembles more a theory of order (Theorem 3) than a set theory, and therefore, while supplying an interesting example for model theory it does not shed any new light on set theory. We feel that this example demonstrates that by generalizing to model companions one may lose the interesting relations between a theory and its model completion.

We deal with a language with a single binary relation $\in$. In a given model we say that there is an *$\in$-chain* leading from b to c if for some $a_1, \cdots, a_n$ $(0 \leq n)$ $b \in a_1 \in \cdots \in a_n \in c$. A *closed $\in$-chain* is a chain leading from some element to itself.

First we define the theory S:

$$S = \{\forall x_1 \cdots x_n(x_1 \notin x_2 \vee \cdots \vee x_{n-1} \notin x_n \vee x_n \notin x_1) | 1 \leq n < \omega\}.$$

S claims that there are no closed $\in$-chains. It is a universal theory (all the axioms are universal sentences) and $S \subset$ ZF by the axiom of regularity. We want to show that S is the universal part of ZF. We show more:

Theorem 1. *If ψ is a universal sentence then either $S \vdash \psi$ or $ZF \vdash \neg\psi$.*

Proof. We assume that not $S \vdash \psi$ and show that ZF $\vdash \neg\psi$. Let ψ be $\forall x_1 \cdots x_n \phi(\bar{x})$ where ϕ is quantifier free. We can assume that ϕ is the disjunction of atomic formulas or their negations: it is at least a conjunction $\phi_1 \wedge \cdots \wedge \phi_k$ of such formulas and if not $S \vdash \forall\bar{x}\phi$ then for some $i \leq k$ not $S \vdash \forall\bar{x}\phi_i(\bar{x})$. If we have shown that ZF $\vdash \neg\forall\bar{x}\phi_i$ then ZF $\vdash \neg\forall\bar{x}\phi$.

Thus we assume that $\phi(\bar{x}) = F_1(\bar{x}) \vee \cdots \vee F_k(\bar{x})$ where every F_i is of

Received by the editors March 29, 1974.
AMS (MOS) subject classifications (1970). Primary 02H05.
Key words and phrases. Model companion.

the form $x_\alpha \in x_\beta$ or $x_\alpha \notin x_\beta$ or $x_\alpha \neq x_\beta$ (we dispose of the case $x_\alpha = x_\beta$ by identifying the variables x_α and x_β). Let $B_i(\bar{x})$ be $\neg F_i(\bar{x})$ for $i = 1, \cdots, k$ and put $H = \{B_i \mid i \leq k\}$. We show that in every model of ZF one can find elements $a_1, \cdots, a_n$ which satisfy all the formulas in H so that ZF $\vdash \neg\psi$.

Let $M \models$ ZF be given and put $X = \{x_1, \cdots, x_n\}$, and let $X_1 = \{x_{j_1}, \cdots, x_{j_s}\}$ be the subset of X of elements x such that no $B \in H$ is of the form $y \in x$. We choose in M elements $a_1, \cdots, a_s, a_{s+1} \cdots a_n$ which are pairwise disjoint and map $x_{j_i} \to a_i$ $(i = 1, \cdots, s)$ (from now on we denote by a_x the element corresponding to x).

Next we let A_1 be the set $\{a_1, \cdots, a_s\}$ and put $X_2 = \{x \in X - X_1 \mid$ if "$y \in x$" is in H then $y \in X_1\}$. To every $x \in X_2$ we relate a_x:

$$a_x = \{a_y \mid \text{"} y \in x \text{" is in } H\} \cup \{b_x\}$$

where b_x is one of the elements $a_{s+1} \cdots a_n$ added to make sets different even if their extension on A_1 is the same.

We proceed in the same way (A_i being sets of elements in $A_0, \cdots, A_{i-1}$) until we get $X_k = \varnothing$. We claim that $X = X_1 \cup \cdots \cup X_{k-1}$. If not then in $Y = X - (X_1 \cup \cdots \cup X_{k-1})$ every element contains some other element of Y (else it would be in X_i for some $i \leq k$). As Y is finite this is possible only if there are $z_1, \cdots, z_q \in Y$ such that $z_1 \in z_2, z_2 \in z_3 \cdots$ and $z_q \in z_1$ are all in H which contradicts the assumption that $\neg \forall \bar{x} \phi(\bar{x})$ is consistent with S.

It is now easy to check that the elements chosen above satisfy $\phi(\bar{x})$. Q.E.D.

Corollaries.

(1) *$ZF_\forall = S$ (all universal consequences of ZF follow already from S).*

(2) *Every model of S can be extended to a model of ZF.*

(3) *If $ZF \subset T$ then every model of ZF can be extended to a model of T.*

(4) *ZF has the* **joint embedding property**—*any two models of ZF can be simultaneously embedded in a third one.*

Proofs. (1) follows immediately from Theorem 1 and implies (2) by 3.11 of [3]. As ZF $\vdash \phi$ or ZF $\vdash \neg\phi$ for every universal sentence the same is true if we replace ZF by any stronger theory T. Hence $T_\forall = S$ and (3) follows again by 3.11 of [3]. (Indeed, reading through the proof of Theorem 1 it is clear that under slight modifications we may replace ZF by any theory T which includes S and such that T implies that for any n elements there is a "set" containing the first r ones and not the rest $(0 \leq r \leq n)$. Thus for every such T we have $T_\forall = S$ and T is mutually model consistent with ZF.) Finally, as ZF $\vdash \phi$ or ZF $\vdash \neg\phi$ if ϕ is universal we have ZF $\vdash \phi$ or ZF $\vdash \psi$ whenever ZF $\vdash \phi \wedge \psi$ and ϕ and ψ are universal. This implies (4) by 4.2 of [3]. Q.E.D.

Next we let Σ be the class of models of S (or the class of submodels of models of ZF which is the same by Corollary 2). Let $\mathfrak{E}$ be the class of existentially complete models in Σ. We are going to show that $\mathfrak{E}$ is an elementary class. We denote by S^* the theory $S \cup S_1 \cup S_2$ where S_1 is the sentence

$$\forall xy\, \exists z(x = y \vee x \in z \in y \vee y \in z \in x)$$

and S_2 is the set of sentences

$$\{\forall x_1 \cdots x_n \exists y(x_1 \in x_2 \in \cdots \in x_n \to$$

$$[x_1\delta_1 y \wedge \cdots \wedge x_r\delta_r y \wedge y\delta_{r+1}x_{r+1} \wedge \cdots \wedge y\delta_n x_n])|0 \leq r \leq n,\ n < \omega\}$$

where δ_i is either $\in$ or $\notin$.

Theorem 2. *If $E \in \mathfrak{E}$ then $E \models S^*$.*

Proof. Let E be a model in $\mathfrak{E}$. Clearly $E \models S$. We show first that $E \models S_1$. Let $a, b \in E$ be given so that $a \neq b$. As $E \models S$ there is no $\in$-chain leading from a to b or there is none leading from b to a. Without loss of generality we assume the last one. We define a model M obtained from E by adding a single new element c and defining $a \in c$ and $c \in b$ (for no other element d of E $d \in c$ or $c \in d$). We claim that $M \models S$. If not then there is a closed $\in$-chain in M which must contain c preceded by a and followed by b. Hence $a_1 \in \cdots \in a \in c \in b \cdots \in a_1$. But this is impossible if no $\in$-chain leads from b to a. Thus we have a model M of S which extends E and has an element c such that $a \in c \in b$. As E is existentially complete such an element can be found in E.

Next we show that $E \models S_2$. Let ϕ be one of the sentences of S_2 and let $a_1 \in a_2 \cdots \in a_n$ be elements of E. We add a new element c to E and obtain a model M by defining $d \in c$ iff d is a_i for some i such that ϕ claims $x_i \in y$, and $c \in d$ iff d is a_i for some i such that ϕ claims $y \in x_i$. Then for $a_1 \cdots a_n$ we have an element y as required by ϕ in M. We show that $M \models S$. If not then we have an $\in$-chain which must contain c preceeded by some a_i and followed by some a_j such that $i < j$. $d_1 \in \cdots \in a_i \in c \in a_j \in \cdots \in d_1$ but then

$$E \models d_1 \in \cdots \in a_i \in a_{i+1} \cdots \in a_j \in \cdots \in d_1$$

which is impossible as $E \models S$.

From the existential completeness of E it now follows that an element like c can be found in E. Hence $E \models \phi$ and $E \models S_2$. Q.E.D.

Theorem 3 emphasizes the difference between ZF and its model companion S^* (see Theorem 5). We begin with it because it simplifies the proof of the converse of Theorem 2.

Definition. Let M be a model of S^*. We write $a < b$ for $\exists x(a \in x \wedge x \in b)$.

Theorem 3. *If $M \models S^*$ then $<$ is a dense linear order (with no end elements). Also: if $a \in b$ then $a < b$.*

Proof. From S we get $\neg(a < a)$. If $a < b$ and $b < c$, S excludes the possibilities $a = c$ and $c < a$ so that by S_1 $a < c$. Therefore we have a partial order which is total by S_1. Assume now that $a \in b$, then by S, $a \neq b$ and not $b < a$ so that $a < b$.

Next assume that $a < b$ so that $a \in c \in b$ for some c. By the remark just made $a < c < b$ so that the order is dense. Finally, given an element b, S_2 implies that there is an element c such that $c \in b$ (and therefore $c < b$). Therefore b is not minimal. A similar argument shows that b is not maximal. Q.E.D.

We strengthen slightly the property S_2.

Lemma. *Let M be a model of S^* and $a_1 < a_2 < \cdots < a_n$ elements of M. Then for every formula*

$$\phi(x) = a_1\delta_1 x \wedge \cdots \wedge a_r\delta_r x \wedge x\delta_{r+1}a_{r+1} \wedge \cdots \wedge x\delta_n a_n$$

we can find a solution $c \in M$ such that $a_r < c < a_{r+1}$.

Proof. By the definition of the order and its density the sequence may be extended to an $\in$-chian $a_1 \in b_1 \cdots \in b_n \in a_n$ such that between a_r and a_{r+1} there are two elements: $a_r \in b_r \in b_{r+1} \in a_{r+1}$. By S_2 we get an element $d \in M$ which satisfies $\phi(d) \wedge b_r \in d \wedge d \in b_{r+1}$. Q.E.D.

Theorem 4. *If $M \models S^*$ then M is existentially complete.*

Proof. Let E be an existentially complete model extending M. For any sequence $c_1, \cdots, c_n \in E$ we denote by $D(c_1 \cdots c_n)$ the conjunction of the formulas $c_i \in c_j$ and $c_i \notin c_j$ according to what the case is in E. We show that for every $a_1, \cdots, a_k \in M$ and $b_1, \cdots, b_m \in E$ there are elements $b'_1, \cdots, b'_m$ in M such that ,

$$D(a_1, \cdots, a_k, b_1, \cdots, b_m) = D(a_1, \cdots, a_k, b'_1, \cdots, b'_m).$$

It is easy to see that this implies that every existential formula with parameters of M that holds in E holds also in M, and as E is existentially complete so also is M. So let $a_1, \cdots, a_k, b_1, \cdots, b_m$ be given. We assume that $a_1 < \cdots < a_k$ in the order of M. As $E \models S^*$, E is also ordered and clearly $a_1 < \cdots < a_k$ also in E. We may assume that $b_1 < \cdots < b_m$ and that $c_1 < \cdots < c_{k+m}$ is the order of all the elements above. We have

4.1. *If "$c_i \in c_j$" is in $D(c_1 \cdots c_{k+m})$ then $c_i < c_j$.*

Thus $D(a_1, \cdots, a_k, b_1)$ is of the form $\phi(b_1)$ for a formula $\phi(x)$ as in the Lemma, and we can find an element $b'_1 \in M$ for which $D(a_1, \cdots, a_k, b_1) = D(a_1, \cdots, a_k, b'_1)$ and so that b'_1 lies at the same place among $a_1 \cdots a_k$ as b_1 does. From this and 4.1 it follows that $D(a_1, \cdots, a_k, b'_1, b_2)$ is of the form $\phi(b_2)$ for some $\phi(x)$ as in the Lemma and we get an element $b'_2 \in M$ such that $D(a_1, \cdots, a_k, b_1, b_2) = D(a_1, \cdots, a_k, b'_1, b'_2)$ and such that b'_2 is ordered among $a_1, \cdots, a_k, b'_1$ similar to how b_2 is ordered among $a_1, \cdots, a_k, b_1$. Continuing with the same argument we obtain $b'_1, \cdots, b'_m$ as required. Q.E.D.

Theorem 5. *S^* is the model companion of ZF.*

Proof. By Theorems 3 and 4, $\mathcal{E}$ is the class of models of S^*, so clearly every model of ZF can be extended to a model of S^* and vice versa (i.e. ZF and S^* are mutually model consistent). As every model of S^* is existentially complete S^* is model complete by Robinson's test [2, p. 191]. Therefore, S^* is the model companion of ZF. Q.E.D.

Corollary. *S^* is the (finite and infinite) forcing companion of ZF.*

By [1] and [3] this is always the case with theories that have model companions. We repeat it here to emphasize that straightforward model theoretic forcing does not yield much of P. J. Cohen's theory.

We conclude with the following remark: While ZF has the joint embedding property (Corollary 4) it violates in a strong sense the amalgamation property. If $M \models$ ZF then there are models M_1 and M_2 of ZF such that $M \subset M_1$, $M \subset M_2$ and there is no way to embed M_1 and M_2 over M in a third model. This is shown by choosing elements $a, b \in M$ such that no $\in$-chain leads from either one to the other and constructing models M_1 with $a \in c_1 \in b$ and M_2 with $b \in c_2 \in a$.

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