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NOTE ON RINGS OF FINITE REPRESENTATION TYPE AND DECOMPOSITIONS OF MODULES

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ABSTRACT. Tachikawa has shown that if a ring Λ is of finite representation type, then each of its left and right modules has a decomposition that complements direct summands. We show that the converse is also true.

Anderson and Fuller [1] posed the problem of determining over which rings does every module have a *decomposition* $M = \bigoplus_A M_\alpha$ that *complements direct summands* in the sense that whenever K is a direct summand of M, $M = K \oplus (\bigoplus_B M_\beta)$ for some $B \subset A$. In response, Tachikawa [6] has proved that the modules over a ring of finite representation type have such decompositions. We recall that an artin ring is of finite representation type if it has only a finite number of finitely generated indecomposable left modules. The purpose of this note is to use the results of [1]–[5] to show that the converse of Tachikawa's result is also true.

Auslander [3] says that a family of R-homomorphisms is *noetherian* if given a sequence

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow \cdots$$

in the family, with $f_i \circ \cdots \circ f_1 f_0 \neq 0$ for all i, there is an integer n such that f_k is an isomorphism for all $k \geq n$, and that the family is *conoetherian* in case given any sequence

$$\cdots \longrightarrow M_2 \xrightarrow{f_1} M_1 \xrightarrow{f_0} M_0$$

with $f_0 f_1 \cdots f_i \neq 0$ for all i, there is an integer n such that f_k is an isomorphism for all $k \geq n$. The result we seek follows from the following result of Auslander

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[3, Theorem 3.1]. *If Λ is a left artin ring over which the family of monomorphisms between finitely generated indecomposable left modules is noetherian and the family of epimorphisms between finitely generated indecomposable left modules is conoetherian then Λ is a ring of finite representation type*;

the following version of Harada and Sai's

[5, Lemma 9]. *Let $\mathfrak{F}$ be a family of finitely generated modules. If every $M = \bigoplus_A M_\alpha$ with $M_\alpha \in \mathfrak{F}$ complements direct summands then the family of homomorphisms between the members of $\mathfrak{F}$ is noetherian*;

and the

Proposition. *Let Λ be an artin ring. If the family of homomorphisms between finitely generated indecomposable right Λ-modules is noetherian then the family of epimorphisms between finitely generated indecomposable left Λ-modules is conoetherian.*

Proof. The proof uses the Auslander-Bridger transpose [4] and some of its properties which can be found in [2] and [4] or derived by standard diagram chasing techniques. Let $(\ )^* = \mathrm{Hom}_\Lambda(\_, \Lambda)$. Let M be a finitely generated left Λ-module that contains no nonzero projective direct summands, and let $P_1 \xrightarrow{d} P_0 \to M \to 0$ be a minimal projective resolution of M. Then the transpose of M is a right Λ-module $\mathrm{Tr}(M)$ such that

$$P_0^* \xrightarrow{d^*} P_1^* \to \mathrm{Tr}(M) \to 0$$

is exact. $\mathrm{Tr}(M)$ is also finitely generated with no nonzero projective direct summands, and M is indecomposable iff $\mathrm{Tr}(M)$ is. If N is another Λ-module with no projective direct summands and $f: M \to N$ is a homomorphism, there is a homomorphism $\hat{f}: \mathrm{Tr}(N) \to \mathrm{Tr}(M)$ satisfying:

(1) if $0 \neq f_0 f_1 \cdots f_i$ *then* $\hat{f}_i \cdots \hat{f}_1 \hat{f}_0 \neq 0$;

(2) f is an isomorphism iff $\hat{f}$ is an isomorphism.

Thus a nonterminating sequence of proper epimorphisms between finitely generated indecomposable (necessarily nonprojective) left Λ-modules

$$\cdots \to M_2 \xrightarrow{f_1} M_1 \xrightarrow{f_0} M_0$$

yields a sequence

$$\mathrm{Tr}(M_0) \xrightarrow{\hat{f}_0} \mathrm{Tr}(M_1) \xrightarrow{\hat{f}_1} \mathrm{Tr}(M_2) \to \cdots$$

such that each $\mathrm{Tr}(M_i)$ is a finitely generated and indecomposable right Λ-module, $\hat{f}_i \cdots \hat{f}_1 \hat{f}_0 \neq 0$ for all i, and no $\hat{f}_k$ is an isomorphism. So the Proposition is proved.

According to [1, Corollary 9], if Λ is a ring whose (projective and injective left) modules all have decompositions that complement direct summands, then Λ is (left) artin. Thus the above results yield

Theorem. *A ring Λ is of finite representation type if (and only if) each of its left and its right modules has a decomposition that complements direct summands.*

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