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AN EVERYWHERE DIVERGENT FOURIER-WALSH SERIES OF THE CLASS $L(\log^+\log^+L)^{1-\epsilon}$ [1]

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ABSTRACT. Let Φ be a function satisfying (a) $\Phi(t) \geq 0$, convex and increasing; (b) $\Phi(t^{1/2})$ is a concave function of t, $0 \leq t < \infty$; and (c) $\Phi(t) = o(t \log \log t)$ as $t \to \infty$. We construct a function in the class

$$\Phi(L) = \{f \in L(0, 1): \textstyle\int_0^1 \Phi(|f(x)|)dx < \infty\}$$

whose Fourier-Walsh series diverges everywhere.

It is known that there exists a function in the class $L(\log^+\log^+L)^{1-\epsilon}$ for $\epsilon > 0$ whose trigonometric series diverges almost everywhere [1]. Let Φ be a function satisfying

(a) $\Phi(t) \geq 0$, convex and increasing in $0 \leq t < \infty$,

(b) $\Phi(t^{1/2})$ is a concave function of t, $0 \leq t < \infty$, and

(c) $\Phi(t) = o(t \log \log t)$ as $t \to \infty$.

For the Walsh system, we will construct a function in the class

$$\Phi(L) = \left\{f \in L(0, 1): \int_0^1 \Phi(|f(x)|)dx < \infty\right\}$$

whose Fourier-Walsh series diverges everywhere by refining Stein's construction [3] of a function in $L(0, 1)$ with almost everywhere divergent Fourier-Walsh series.

We recall the definition of the Walsh system in the Paley enumeration. The Rademacher functions $r_n(x)$ are defined by

$$(1)\qquad \begin{aligned} &r_0(x) = 1 \quad (0 \leq x < 1/2), && r_0(x) = -1 \quad (1/2 \leq x < 1), \\ &r_0(x+1) = r_0(x), && r_n(x) = r_0(2^n x) \quad (n = 1, 2, \cdots). \end{aligned}$$

For each positive integer n, there is a unique representation of the form $n = \sum_{j=0}^{\infty} \epsilon_j 2^j$, where $\epsilon_j = 0$ or 1. The Walsh functions in the Paley enumeration are then given by

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$$w_0(x) = 1, \qquad w_n(x) = \prod_{j=0}^{\infty} [r_j(x)]^{\epsilon_j}. \tag{2}$$

Let x be any real number in $(0, 1)$. Then we have a unique representation of the form $x = \sum_{n=1}^{\infty} x_n 2^{-n}$ with infinitely many $x_n \neq 0$, where $x_n = 0$ or 1.

We define

$$x \dotplus y = \sum_{n=1}^{\infty} |x_n - y_n| 2^{-n} \tag{3}$$

where $x = \sum_{n=1}^{\infty} x_n 2^{-n}$, $y = \sum_{n=1}^{\infty} y_n 2^{-n}$, $x_n, y_n = 0$ or 1 and the operation '$\dotplus$' is called dyadic addition (see Fine [2]).

We denote the Dirichlet kernel and the partial sum of Fourier series of $f(x)$ with respect to the Walsh functions in the Paley enumeration by

$$\begin{aligned} D_n(x) &= \sum_{j=0}^{n-1} w_j(x), \\ S_n f(x) &= \sum_{j=0}^{n-1} C_j(f) w_j(x) = \int_0^1 f(t) D_n(x \dotplus t)\,dt, \end{aligned} \tag{4}$$

where $C_j = C_j(f) = \int_0^1 f(t) w_j(t)\,dt$ is the jth Fourier coefficient of f.

The Lebesgue constant L_n is given by

$$L_n = \int_0^1 |D_n(t)|\,dt. \tag{5}$$

It is well known (see Fine [2]) that

$$\limsup_{n \to \infty} \left(\frac{L_n}{\log n} \right) \geq \alpha > 0. \tag{6}$$

An interval I with the length 2^{-n} is called a *dyadic interval* if the $(n-1)$th Rademacher function $r_{n-1}(t)$ is constant on I.

First of all we want to prove the following lemma, from which our main theorem follows. A part of the proof of this lemma will use a technique of E. M. Stein in [3].

Lemma. *For any fixed positive integer n, there exists a set E_n such that*

(i) $m(E_n) = 2^{-2^N}$, *where* $2^{N-1} \leq n < 2^N$,

(ii) $C_k(\chi_{E_n}) = \int_0^1 \chi_{E_n}(t) w_k(t)\,dt = 0$ *if* $0 < k < 2^N$ *or* $k \geq 2^{N+2^N}$,

(iii) $M\chi_{E_n}(x) = \sup_{n>1} |S_n \chi_{E_n}(x)| \geq \frac{1}{2} L_n m(E_n)$,

where $m(A)$ and χ_A denote the Lebesgue measure and the characteristic function of the set A respectively.

Proof. Let $I_j = [(j-1)2^{-N}, j2^{-N})$ $(j = 1, 2, \cdots, 2^N)$. We will choose dyadic intervals d_j such that $d_j \subset I_j$ and $m(d_j) = 2^{-(N+2^N)}$ for all $j = 1, 2, \cdots, 2^N$, and put $E_n = \bigcup_{j=1}^{2^N} d_j$.

Then we get

$$m(E_n) = \sum_{j=1}^{2^N} m(d_j) = 2^{-2^N}. \tag{7}$$

We note that for any k with $0 < k < 2^N$

$$m\{t \in E_n;\ w_k(t) = 1\} = m\{t \in E_n;\ w_k(t) = -1\}$$

and for any $k \geq 2^{N+2^N}$

$$m\{t \in d_j;\ w_k(t) = 1\} = m\{t \in d_j;\ w_k(t) = -1\}$$

for all $j = 1, 2, \cdots, 2^N$. Hence, we obtain

$$C_k(\chi_{E_n}) = 0 \qquad \text{if } 0 < k < 2^N \text{ or } k \geq 2^{N+2^N}. \tag{8}$$

It remains to choose the dyadic intervals d_j so that (iii) is satisfied.

We note that for $n < 2^N$, $D_n(t) = \sum_{j=0}^{n-1} w_j(t)$ is constant on I_i for each $i = 1, 2, \cdots, 2^N$, and hence $D_n(x \dotplus t)$ is constant as x and t vary over I_i and I_j respectively. Let $D_n(I_i \dotplus I_j)$ denote the value of $D_n(x \dotplus t)$ for $x \in I_i$ and $t \in I_j$, and

$$\sigma(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases} \tag{9}$$

Consider the 2^N-tuples R_k $(1 \leq k \leq 2^N)$ such that

$$R_k = (\sigma(D_n(I_k \dotplus I_1)), \sigma(D_n(I_k \dotplus I_2)), \cdots, \sigma(D_n(I_k \dotplus I_{2^N}))). \tag{10}$$

We now define the dyadic interval d_j by

$$d_j = \left[(j-1)2^{-N} + 2^{-N} \sum_{i=1}^{2^N} \epsilon_{ji} 2^{-i},\ (j-1)2^{-N} + 2^{-N} \sum_{i=1}^{2^N} \epsilon_{ji} 2^{-i} + 2^{-(N+2^N)} \right) \tag{11}$$

where ϵ_{ji} $(1 \leq i \leq 2^N,\ 1 \leq j \leq 2^N)$ is either 0 or 1 and

$$(-1)^{\epsilon_{ji}} = \sigma(D_n(I_i \dotplus I_j)). \tag{12}$$

Hence, for all $t \in d_j$ $(1 \leq j \leq 2^N)$

$$w_{2^{N+i-1}}(t) D_n(I_i \dotplus I_j) = r_{N+i-1}(t) D_n(I_i \dotplus I_j) = (-1)^{\epsilon_{ji}} D_n(I_i \dotplus I_j) \geq 0 \tag{13}$$

for each i with $1 \leq i \leq 2^N$.

Now we set $E_n = \bigcup_{j=1}^{2^N} d_j$ and it remains to show that χE_n satisfies condition (iii).

For any fixed $x \in [0, 1)$, there exists a unique k such that $x \in I_k$, and we set

(14) $$n_{kx} = n + 2^{N+k-1} \qquad (2^{N-1} \leq n < 2^N,\ n \text{ fixed}).$$

We again note that

(15) $$D_n(x \dotplus t) w_{2^{N+k-1}}(t) \geq 0$$

for all $t \in d_j$ and $j = 1, 2, \cdots, 2^N$. Since $D_n(x \dotplus t)$ is constant on each $I_i\ (I \leq i \leq 2^N)$ and $m(E_n) = 2^{-2^N}$, we obtain, by applying (15),

$$|S_{n_{kx}} \chi E_n(x) - S_{2^{N+k-1}} \chi E_n(x)| = \left| \int_0^1 \chi E_n(t) w_{2^{N+k-1}}(t) D_n(x \dotplus t)\,dt \right|$$

(16) $$= \left| \sum_{j=1}^{2^N} \int_{d_j} w_{2^{N+k-1}}(t) D_n(x \dotplus t)\,dt \right| = \sum_{j=1}^{2^N} \int_{d_j} |D_n(x \dotplus t)|\,dt$$

$$= 2^{-2^N} \sum_{j=1}^{2^N} \int_{I_j} |D_n(x \dotplus t)|\,dt = m(E_n) \int_0^1 |D_n(x \dotplus t)|\,dt = m(E_n) \cdot L_n.$$

Thus, (16) implies $M\chi E_n(x) \geq \frac{1}{2} L_n m(E_n)$. The lemma is proved.

Now we are ready to prove the following theorem:

Theorem. *Let Φ be a function satisfying conditions* (a), (b) *and* (c). *Then there exists a function $f \in \Phi(L(0, 1))$ such that $S_n f(x)$ diverges everywhere.*

Proof. If we note (6) and properties of the function Φ, we may choose a sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers satisfying the following conditions:

(α) there is a constant $A > 0$ such that $L_{n_j} \geq A \log n_j$,

(β) $N_{j+1} \geq N_j + 2^{N_j}$, and

(γ) $\Phi(\alpha_j) \leq j^{-2} \alpha_j (\log \log \alpha_j)$,

where $2^{N_j - 1} \leq n_j < 2^{N_j}$, $\alpha_j = 1/(\log n_j) m(E_{n_j})$, $m(E_{n_j}) = 2^{-2^{N_j}}$ and the sets E_{n_j} are the same as in the lemma. It is easy to see that the sequence $\{\alpha_n\}_{n \geq 1}$ is a lacunary sequence and there exists a constant C such that

(17) $$\sum_{j=1}^{n} \alpha_j \leq C \alpha_n.$$

Let f be the measurable function defined by

$$f(x) = \sum_{j=1}^{\infty} \alpha_j \chi_{E_{n_j}}(x). \tag{18}$$

From the properties of Φ and (17) we get

$$\Phi\left(\sum_{j=1}^{\infty} \alpha_j \chi_{E_{n_j}}(x)\right) \le C \sum_{j=1}^{\infty} \Phi(\alpha_j)\chi_{E_{n_j}}(x). \tag{19}$$

In fact, if x does not belong to $\bigcup_{j=1}^{\infty} E_{n_j}$ or x belongs to infinitely many E_{n_j}'s then both sides of (19) are equal to 0 or ∞ respectively, and if x belongs to finitely many E_{n_j}'s then

$$\Phi\left(\sum_{j=1}^{\infty} \alpha_j \chi_{E_{n_j}}(x)\right) = \Phi\left(\sum_{j=1}^{k} \alpha_j \chi_{E_{n_j}}(x)\right) \le \Phi\left(\sum_{j=1}^{k} \alpha_j\right) \chi_{E_{n_k}}(x)$$

$$\le \Phi(C\alpha_k)\chi_{E_{n_k}}(x) \le C \sum_{j=1}^{\infty} \Phi(\alpha_j)\chi_{E_{n_j}}(x)$$

where $k = \max\{j;\ x \in E_{n_j}\} < \infty$. Hence, we have

$$\int_0^1 \Phi(f(x))dx \le C \sum_{j=1}^{\infty} \Phi(\alpha_j) m(E_{n_j}) \le C \sum_{j=1}^{\infty} \frac{1}{j^2} \alpha_j (\log\log \alpha_j) m(E_{n_j}) < \infty. \tag{20}$$

This implies $f \in \Phi(L)$.

Now it remains to show that $S_n f(x)$ diverges everywhere. Let x be a fixed point in $[0, 1)$.

For each positive integer k, (14) and (16) imply that there exists a positive integer n_{kx} such that

$$n_{kx} = n_k + 2^{N_{kx}} \quad \text{with } N_k \le N_{kx} < N_k + 2^{N_k}, \tag{21}$$

and

$$|S_{n_{kx}} \chi_{E_{n_k}}(x) - S_{2^{N_{kx}}} \chi_{E_{n_k}}(x)| = L_{n_k} m(E_{n_k}). \tag{22}$$

If $j \ne k$, we obtain

$$S_{n_{kx}} \chi_{E_{n_j}}(x) - S_{2^{N_{kx}}} \chi_{E_{n_j}}(x) = \sum_{i=2^{N_{kx}}}^{n_{kx}-1} C_i(\chi_{E_{n_j}}) w_i(x) = 0 \tag{23}$$

since part (ii) of the lemma implies $C_i(\chi_{E_{n_j}}) = 0$ if $2^{N_k} \le i < 2^{N_k+1}$. A combination of (22), (23), (18) and (α) gives

$$|S_{n_{kx}} f(x) - S_{2^{N_{kx}}} f(x)| = \alpha_k |S_{n_{kx}} \chi_{E_{n_k}}(x) - S_{2^{N_{kx}}} \chi_{E_{n_k}}(x)|$$

$$= \alpha_k L_{n_k} m(E_{n_k}) = L_{n_k} / \log n_k \ge A > 0.$$

We finally get

$$\limsup_{m,n\to\infty} |S_m f(x) - S_n f(x)| \geq A > 0 \tag{24}$$

for all $x \in [0, 1)$, that is, the Fourier-Walsh series of $f \in \Phi(L)$ diverges everywhere.

Remark. A theorem in E. M. Stein [3, Theorem 3] implies that if for every $f \in \Phi(L)$

$$m\left\{x \in (0, 1): \limsup_{n\to\infty} |S_n f(x)| < \infty\right\} > 0 \tag{25}$$

then there exists an absolute constant A such that for any $y > 0$

$$m\left\{x \in (0, 1): \sup_{n\geq 1} |S_n f(x)| > y\right\} \leq \int_0^1 \Phi\left(\frac{A}{y}|f(x)|\right)dx. \tag{26}$$

We may apply this theorem to prove the existence of a function in the class $\Phi(L)$ whose Fourier-Walsh series diverges almost everywhere.

In fact, let $f(x) = \chi_{E_n}(x)$ and $y_n = \frac{1}{2} L_n m(E_n)$, where the set E_n is defined in the lemma. Then part (iii) of the Lemma implies

$$m\{x \in (0, 1): M\chi_{E_n}(x) > y_n\} = 1$$

for all positive integers n, but for ϵ, $0 < \epsilon < 1$,

$$\int_0^1 \Phi\left(\frac{A\chi_{E_n}(x)}{y_n}\right)dx \leq \epsilon < 1$$

for all sufficiently large n where the constant A is as same as in the inequality (26).

Thus, our theorem for the almost everywhere divergence follows.

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