PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, July 1975

BINOMIAL λ-RINGS AND A TOPOLOGICAL COROLLARY

DONALD KNUTSON

ABSTRACT. A result in the algebra of binomial λ-rings implies that triviality of Adams operations in topological K-theory or in group representation theory implies triviality of the whole K-ring.

A λ-ring B is called *binomial* if for each $b \in B$ and integer $n \geq 0$, $n\lambda^n(b) = b\lambda^{n-1}(b-1)$. For example, the integers $\mathbf{Z}$, with $\lambda^n(m) = \binom{m}{n}$, the binomial coefficient, satisfies this condition. An equivalent definition is that the Adams operations Ψ^n, $n \geq 1$, are all the identity map: $\Psi^n(x) = x$, all x. (Note that by [1, pp. 48–49], a pre-λ-ring which is binomial must be a λ-ring.)

In [1] we noted that for a finite group G, the representation ring $R(G)$ can only be binomial when G is the trivial group. The proof involved explicit identification of the Adams operations. However, the fact is much more basic as the following theorem shows.

Theorem. *Let B be a binomial λ-ring. Suppose there is a subset $A \subset B$ satisfying* (a) *A generates B as an abelian group,* (b) *for each $x \in A$, $\lambda_t(x)$ is a polynomial (whose degree will be called the dimension of x),* (c) *for each $x \in A$ and integer $n \geq 1$, $\lambda^n(x) \in A$, and* (d) *for each $x \in A$ of dimension 1, there is an element $y \in B$ with $xy = 1$. Then B is isomorphic to $\mathbf{Z}$.*

Proof. Let $x \in A$ be of dimension n. Then $\lambda^n(x)$ is of dimension one. (**Proof.** By the splitting principle we can assume B is extended to a perhaps larger λ-ring in which x is expressible as the sum of n one-dimensional elements. Then $\lambda^n(x)$ is the product of these elements and hence one-dimensional.)

Let $a = \lambda^n(x)$. Since a is of dimension one, $\Psi^n(a) = a^n$, $n \geq 1$. But since B is binomial $\Psi^n(a) = a$. Hence $a = a^2$. By hypotheses, $a \in A$ and there is an element b with $ab = 1$. Hence $1 = ab = a^2b = a$.

Now $\lambda_t(x) = \lambda_t(x-1)\lambda_t(1) = \lambda_t(x-1)(1+t)$, so $\lambda^n(x) = \lambda^{n-1}(x-1) + \lambda^n(x-1)$, in any λ-ring. Hence in our case,

Received by the editors December 6, 1973 and, in revised form, May 24, 1974.
AMS (MOS) subject classifications (1970). Primary 20C15, 55G25.

$$x = x\lambda^n(x) = x\lambda^{n-1}(x-1) + x\lambda^n(x-1) = n\lambda^n(x) + (n+1)\lambda^{n+1}(x) = n + 0 = n.$$

Thus the only elements in the subset A are positive integers. Since these generate B, $B = \mathbf{Z}$. Q.E.D.

Corollary 1. *Let $R(G)$ be the representation ring of a finite group G. Suppose $R(G)$ is binomial. Then $G = \{e\}$.*

Proof. Let A be the set of actual representations. By the above, $R(G) = \mathbf{Z}$. But the rank of $R(G)$ as abelian group is the number of conjugacy classes of elements. Since no element is conjugate to the identity, this forces G to be trivial. Q.E.D.

Corollary 2. *Let $K(X)$ be the Grothendieck ring of vector bundles on a connected compact Hausdorff space. Suppose that all the Adams operations Ψ^n are the identity. Then $K(X) = \mathbf{Z}$.*

Proof. Let A be the set of actual vector bundles on X, whose Grothendieck ring is $K(X)$. Recall that in a connected space, any line bundle L has an inverse L^{-1} with $L \otimes L^{-1} = 1$. Hence the Theorem applies. Q.E.D.

We remark that it is not the case that $K(X) = \mathbf{Z}$ forces a space to be trivial—i.e., contractible. There are nontrivial finite CW complexes with trivial K-theory.

We thank the referee for a helpful suggestion on the original proof of the Theorem.

REFERENCE

1. D. Knutson, *λ-rings and the representation theory of the symmetric group*, Lecture Notes in Math., vol. 308, Springer-Verlag, Berlin and New York, 1973.

DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, BRONX, NEW YORK 10458