REFERENCES

1. P. R. Garabedian, *Partial differential equations with more than two independent variables in the complex domain*, J. Math. Mech. 9 (1960), 241–271. MR 22 #11195.

2. ———, *Partial differential equations*, Wiley, New York, 1964. MR 28 #5247.

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A BOUND-TWO ISOMORPHISM BETWEEN $C(X)$ BANACH SPACES

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ABSTRACT. Nonhomeomorphic compact Hausdorff spaces X and Y and an isomorphism ϕ: $C(X) \to C(Y)$ (onto) are constructed such that $\|\phi\| \|\phi^{-1}\| = 2$. Amir had asked if such a ϕ exists with $\|\phi\| \|\phi^{-1}\| < 3$.

1. **Introduction.** In 1965, D. Amir [1] showed that if $C(X)$ and $C(Y)$ admit an (onto) isomorphism ϕ whose bound $\|\phi\| \|\phi^{-1}\|$ is less than 2, then X and Y are homeomorphic. Here X and Y are compact Hausdorff spaces and $C(X)$, $C(Y)$ are the sup-norm Banach spaces of real-valued continuous functions on X and Y, respectively. In the same paper he constructed nonhomeomorphic X and Y and an isomorphism ϕ: $C(X) \longrightarrow C(Y)$ with bound exactly 3. Amir posed the problem of determining whether or not an isomorphism can exist for nonhomeomorphic X and Y with $\|\phi\| \|\phi^{-1}\| < 3$. See also [5, p. 155]. The purpose of this paper is to exhibit such an example, with $\|\phi\| \|\phi^{-1}\| = 2$.

We note that a similar problem was settled by Camburn for the larger class of sup-norm Banach spaces $C_0(T)$ of complex valued continuous

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functions vanishing at infinity on the locally compact Hausdorff space T. If $C_0(T)$ and $C_0(S)$ admit an isomorphism whose bound is less than 2, then S and T are homeomorphic [2], and there exists an isomorphism ϕ: $C_0(T) \to C(S)$ whose bound is exactly 2 with T not compact and S compact [3]. Amir's original problem, for X and Y both compact, has persisted, however, with Gordon's results [4] suggesting that X and Y must be homeomorphic for $\|\phi\| \, \|\phi^{-1}\| < 3$.

The example below was discovered in the process of deducing topological relationships between X and Y in the presence of an isomorphism ϕ satisfying $\|\phi\| \, \|\phi^{-1}\| \leq 2$. These results, including a new approach to Amir's theorem, involve the second duals $C(X)^{**}$ and $C(Y)^{**}$, and will be published elsewhere.

2. **The example.** Let $I = [0, 1]$ and $a, \bar{a}, b, \bar{b}, c, \bar{c}, d, \bar{d}$ eight homeomorphisms whose domain is I. Let A, B, C, D be four arbitrary compact Hausdorff spaces with distinguished points P_A, P_B, P_C, P_D, respectively. Assume these 12 spaces $a[I], \bar{a}[I], b[I], \ldots, A, B, C, D$ are pairwise disjoint.

Define Y as the disjoint union of $a[I]$, $b[I]$, $c[I]$, $d[I]$, A, B, C, D with the following identifications: $a(0) = b(0)$, $c(0) = d(0)$, $a(1) = P_A$, $b(1) = P_B$, $c(1) = P_C$, $d(1) = P_D$.

Define X to be the disjoint union of $\bar{a}[I]$, $\bar{b}[I]$, $\bar{c}[I]$, $\bar{d}[I]$, A, B, C, D with the identifications: $\bar{a}(0) = \bar{c}(0)$, $\bar{b}(0) = \bar{d}(0)$, $\bar{a}(1) = P_A$, $\bar{b}(1) = P_B$, $\bar{c}(1) = P_C$, and $\bar{d}(1) = P_D$. For suitable choices of A, B, C and D, X and Y are not homeomorphic; e.g., if A and C are circles and B and D singletons. Define ϕ: $C(X) \to C(Y)$ as follows. Let $f \in C(X)$.

$$
\begin{aligned}
\phi(f)(a(t)) &= (1+t)f(\bar{a}(t)) - (1-t)f(\bar{d}(t)),\\
\phi(f)(d(t)) &= (1-t)f(\bar{a}(t)) + (1+t)f(\bar{d}(t)),\\
\phi(f)(b(t)) &= -(1+t)f(\bar{b}(t)) + (1-t)f(\bar{c}(t)),\\
\phi(f)(c(t)) &= (1-t)f(\bar{b}(t)) + (1+t)f(\bar{c}(t)), \quad 0 \leq t \leq 1,\\
\phi(f)(z) &= 2f(z) \qquad \text{if } z \in A \cup C \cup D,\\
&= -2f(z) \quad \text{if } z \in B.
\end{aligned}
$$

On each of the eight spaces $a[I]$, $b[I]$, $c[I]$, $d[I]$, A, B, C and D, $\phi(f)$ is continuous and $\phi(f)$ is consistent with the identifications which define Y from these spaces. Therefore, $\phi(f) \in C(Y)$, and ϕ is well defined. Clearly, ϕ is linear and $\|\phi\| = 2$.

Define $\psi : C(Y) \to C(X)$ as follows. Let $g \in C(Y)$ and

$$D(t) = (1+t)^2 + (1-t)^2 = 2(1+t^2),$$

$$\psi(g)(\bar{a}(t)) = \frac{1+t}{D(t)} g(a(t)) + \frac{(1-t)}{D(t)} g(d(t)),$$

$$\psi(g)(\bar{d}(t)) = -\frac{(1-t)}{D(t)} g(a(t)) + \frac{1+t}{D(t)} g(d(t)),$$

$$\psi(g)(\bar{b}(t)) = -\frac{(1+t)}{D(t)} g(b(t)) + \frac{(1-t)}{D(t)} g(c(t)),$$

$$\psi(g)(\bar{c}(t)) = \frac{(1-t)}{D(t)} g(b(t)) + \frac{(1+t)}{D(t)} g(c(t)), \quad 0 \leq t < 1,$$

$$\begin{aligned} \psi(g)(v) &= \tfrac{1}{2} g(v) \quad \text{if } v \in A \cup C \cup D, \\ &= -\tfrac{1}{2} g(v) \quad \text{if } v \in B. \end{aligned}$$

The function $\psi(g)$ is well defined and continuous; therefore, ψ is defined and, evidently, linear. Since $2/D(t) = 1/(1+t^2) \leq 1$, $\|\psi\| = 1$.

It can be checked that ϕ and ψ are mutually inverse transformations.

REFERENCES

1. Dan Amir, *On isomorphisms of continuous function spaces*, Israel J. Math. 3 (1965), 205–210. MR 34 #596.

2. Michael Cambern, *On isomorphisms with small bound*, Proc. Amer. Math. Soc. 18 (1967), 1062–1066. MR 36 #669.

3. ———, *Isomorphisms of $C_0(Y)$ onto $C(X)$*, Pacific J. Math. 35 (1970), 307–312.

4. Y. Gordon, *On the distance coefficient between isomorphic function spaces*, Israel J. Math. 8 (1970), 391–397. MR 42 #5021.

5. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Lecture Notes in Math., vol. 338, Springer-Verlag, Berlin, 1973.

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