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A MEAN VALUE FORMULA FOR THE SPIN GROUP

LAWRENCE VERNER[1]

ABSTRACT. An adelic mean value formula is proved for two-connected algebraic homogeneous spaces, generalizing Siegel's formula in the geometry of numbers. The case of the spin group acting on the generalized sphere furnishes an example. The procedure consists in applying Galois cohomological techniques to the method of Ono.

Let q be a quadratic form on $\mathbf{C}^n$, and let $X = \{x \in \mathbf{C}^n: q(x) = 1\}$. The group $SO(q) = \{g \in \mathrm{SL}(n, \mathbf{C}): q(gx) = q(x)\}$ acts transitively on X. Applying Witt's theorem to the homogeneous space $(SO(q), X)$, Ono proved a "mean value formula" which is analogous to Siegel's mean value theorem [4]. Our intention is to prove the mean value formula for certain other spaces.

The action of $SO(q)$ on X can be lifted to $\mathrm{Spin}(q)$, the universal covering group of $SO(q)$, as follows. Let $p: \mathrm{Spin}(q) \longrightarrow SO(q)$ be the covering map. For $g \in \mathrm{Spin}(q)$, $x \in X$, set $g^*x = p(g)x$. Then $\mathrm{Spin}(q)$ acts transitively on X but does not satisfy the Witt condition unless q has maximal Witt index. In general we consider an algebraic group G acting transitively on a variety X. Ono has shown that the mean value theorem holds for (G, X) if certain topological conditions are satisfied [3]. A variation of Ono's methods will enable us to prove a theorem which contains the mean value formula for the spin group as a special case.

1. **Statement of the Theorem.** Let G be a connected linear algebraic group defined over a field k. Let X be an algebraic variety defined over k upon which G acts k-rationally. The pair (G, X) is a homogeneous space defined over k if the action is transitive and $X_k \neq \varnothing$.

Theorem. *Let (G, X) be a homogeneous space defined over an algebraic number field k. Suppose X is 2-connected as a complex manifold and G is*

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1-connected as a complex Lie group. Then, for every continuous function f on the adele space X_A with compact support,

$$\int_{G_A/G_k} \sum_{x \in X_k} f(gx)\, d\dot{g} = \int_{X_A} f(x)\, dx.$$

In particular, both sides of the formula are meaningful and finite.

If $n \geq 3$, the group $\mathrm{Spin}(q)$ is 1-connected, and if $n \geq 4$, the "sphere" X is 2-connected (see [3, Exercise II, p. 279]). If $n \geq 5$, then by Hasse's theorem $q(x) = 1$ has a rational solution and so $(\mathrm{Spin}(q), X)$ is a homogeneous space defined over $\mathbf{Q}$ satisfying the conditions of the Theorem.

Our proof of the Theorem requires the conjectures of Weil and Kneser on 1-connected groups, although for the spin group these conjectures are known results.

2. **Canonical measures.** If G is an algebraic group with a finite fundamental group, then there is a canonical Haar measure dg of the adele group G_A, and the measure of the quotient space $\tau(G) = \int_{G_A/G_k} d\dot{g}$ is finite (see [5]). Canonical measures can also be defined on suitable homogeneous spaces.

Lemma 1. *If (G, X) satisfies the conditions of the Theorem, then the isotropy subgroup $G_x = \{g \in G: gx = x\}$, is 1-connected for each $x \in X$.*

Proof. Fix $x \in X$. The map $G \longrightarrow X$, given by $g \longrightarrow gx$, gives rise to an exact sequence.

$$0 \longrightarrow G_x \longrightarrow G \longrightarrow X \longrightarrow 0. \tag{E}$$

This in turn induces a finite covering

$$0 \longrightarrow G_x/(G_x)_0 \longrightarrow G/(G_x)_0 \longrightarrow X \longrightarrow 0,$$

where $(G_x)_0$ denotes the connected component of the identity in G_x. Since X is simply-connected, the covering must be trivial; hence, $G_x/(G_x)_0 = 0$, i.e., G_x is connected.

By the homotopy exact sequence associated with (E) we have $\pi_2(X) \longrightarrow \pi_1(G_x) \longrightarrow \pi_1(G)$. Our hypotheses imply that $0 = \pi_2(X) = \pi_1(G)$, and so G_x is simply-connected.

Proposition. *If (G, X) is a homogeneous space defined over k, satisfying the conditions of the Theorem, then*

(a) *X_k is discrete in X_A;*

(b) *there is a canonical measure* dx *on* X_A;

(c) dx *is* G_A*-invariant.*

Proof. Lemma 1 shows that (G, X) is a "special homogeneous space" in the language of [3]. Theorem 4.1 of [3] therefore applies.

Weil's conjecture states that if G is a 1-connected algebraic group, then $\tau(G) = 1$. Consequently our Theorem gives a formula for the mean value of the function $g \mapsto \Sigma_{x \in X_k} f(gx)$.

3. **Galois cohomology.** Let (G, X) be a homogeneous space defined over a field k. Although G acts transitively on X, G_k need not act transitively on X_k. Instead, we have, for each $\xi \in X_k$, an exact sequence of Galois cohomology sets:

$$0 \longrightarrow G_{\xi,k} \xrightarrow{\iota_k} G_k \xrightarrow{\kappa_k} X_k \xrightarrow{\delta_k} H^1(k, G_\xi) \xrightarrow{\Delta_k} H^1(k, G).$$

Two elements in X_k have the same image in $H^1(k, G_\xi)$ if and only if they are in the same G_k-orbit. The main properties of Galois cohomology can be found in [1].

Let G be an algebraic group defined over an algebraic number field k. The completions $k \longrightarrow k_v$ induce a map

$$\alpha\colon H^1(k, G) \longrightarrow \Pi H^1(k_v, G),$$

the product being taken over all places of k. Kneser has conjectured that α is bijective if G is 1-connected [2]. This is known to hold for all groups without an E_8-factor. In what follows we assume the validity of Kneser's conjecture.

Lemma 2. *Let* (G, X) *satisfy the conditions of the Theorem. Then,* $G_A X_k = X_A$.

Proof. Let $x \in X_A$, and consider the commutative diagram:

$$\begin{tikzcd} X_k \arrow[r, "\delta_k"] \arrow[d, "j"] & \Pi H^1(k, G_\xi) \arrow[r, "\Delta_k"] \arrow[d, "\alpha"] & \Pi H^1(k, G) \arrow[d, "\beta"] \\ \Pi X_{k_v} \arrow[r, "\Pi\delta_v"] & \Pi H^1(k_v, G_\xi) \arrow[r, "\Pi\Delta_v"] & \Pi H^1(k_v, G). \end{tikzcd}$$

We view $x \in \Pi X_{k_v}$. Let $(c_v) = \Pi\delta_v(x)$. Lemma 1 shows that G_ξ is 1-connected and so by Kneser's conjecture α is surjective and we can lift (c_v) to $c \in H^1(k, G_\xi)$. Now

$$\beta \circ \Delta_k(c) = \Pi\Delta_v \circ \alpha(c) = \Pi\Delta_v(c_v) = \Pi\Delta_v \circ \Pi\delta_v(x) = 0.$$

By Kneser's conjecture, β is injective and, hence, $\Delta_k(c) = 0$. By exactness we can write $c = \delta_k(\eta)$, for some $\eta \in X_k$. Then

$$\Pi\delta_v \circ j(\eta) = \alpha \circ \delta_k(\eta) = \alpha(c) = (c_v) = \Pi\delta_v(x);$$

i.e., for each place v, $\delta_v(\eta) = \delta_v(x)$, and so x and η are in the same orbit under G_{k_v}. For each v select $g_v' \in G_{k_v}$ such that $g_v'\eta = x$. For almost all places v, the group of v-integral points, G_{0_v}, acts transitively on X_{0_v} (see [1, p. 161]). Let S be the set of places where this transitivity fails. For each $v \notin S$ choose $h_v \in G_{0_v}$ such that $h_v\eta = x$. Now set

$$g_v = \begin{cases} g_v' & \text{if } v \in S, \\ h_v & \text{if } v \notin S. \end{cases}$$

Then $g = (g_v) \in G_A$, and $x = g\eta \in G_A X_k$ as desired.

Lemma 3. *If* (G, X) *satisfies the conditions of the Theorem, then the map* $J\colon G_k\backslash X_k \to G_A\backslash X_A$, *given by* $J(G_k\xi) = G_A\xi$, *is bijective.*

Proof. Take $G_Ax \in G_A\backslash X_A$. By Lemma 2 we can write $x = g\eta$, with $g \in G_A$, $\eta \in X_k$. Then $G_Ax = G_A\eta = J(G_k\eta)$, so J is surjective. To see that J is injective consider the commutative diagram

$$\begin{array}{ccccc} G_k & \xrightarrow{\kappa_k} & X_k & \xrightarrow{\delta_k} & H^1(k, G_\xi) \\ \downarrow{\scriptstyle j_G} & & \downarrow{\scriptstyle j_X} & & \downarrow{\scriptstyle \alpha} \\ \Pi G_{k_v} & \xrightarrow{\Pi\kappa_v} & \Pi X_{k_v} & \xrightarrow{\Pi\delta_v} & \Pi H^1(k_v, G_\xi). \end{array}$$

$$\begin{aligned} J(G_k\xi) = J(G_k\eta) \quad & ((\xi, \eta \in X_k) \Rightarrow \eta \in G_A\xi) \\ & \Rightarrow j_X(\eta) \in \operatorname{Im}(\Pi\kappa_v) = \operatorname{Ker}(\Pi\delta_v) \\ & \Rightarrow \alpha \circ \delta_k(\eta) = 0 \\ & \Rightarrow \delta_k(\eta) = 0 \quad \text{(Kneser's conjecture)} \\ & \Rightarrow \eta \in \operatorname{Im}(\kappa_k) = G_k\xi \Rightarrow G_k\xi = G_k\eta. \end{aligned}$$

4. **Proof of the Theorem.** Let (G, X) be a homogeneous space defined over k, satisfying the conditions of the Theorem. Let f be a continuous

function on X_A with compact support. For each $\xi \in X_k$ let du_ξ be the canonical Haar measure on $G_{\xi,A}$.

Lemma 4 [3, Lemma 5.1]. *Each orbit $G_A\xi$ is open in X_A.*

Lemma 4 shows that the measure dx on X_A restricts to a measure on G_A.

As an application of Fubini's theorem we have (see [5, Lemma 2.4.2])

$$\int_{G_A/G_k} \sum_{x \in G_k\xi} f(gx)\, d\dot{g} = \int_{G_A\xi} f(x)\, dx \cdot \int_{G_{\xi,A}/G_{\xi,k}} d\dot{u}_\xi = \int_{G_A\xi} f(x)\, dx,$$

by Weil's conjecture applied to G_ξ.

Let D be a fundamental domain for G_k in X_k. Lemma 3 shows that D is also a fundamental domain for G_A in X_A. Hence,

$$\begin{aligned}\int_{G_A/G_k} \sum_{x \in X_k} f(gx)\, d\dot{g} &= \int_{G_A/G_k} \sum_{\xi \in D} \sum_{x \in G_k\xi} f(gx)\, d\dot{g} \\ &= \sum_{\xi \in D} \int_{G_A/G_k} \sum_{x \in G_k\xi} f(gx)\, d\dot{g} \\ &= \sum_{\xi \in D} \int_{G_A\xi} f(x)\, dx = \int_{X_A} f(x)\, dx = \int_{X_A} f(x)\, dx.\end{aligned}$$

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DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, BARUCH COLLEGE, NEW YORK, NEW YORK 10010