

## CONFUSION IN THE GARDEN OF EDEN

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**ABSTRACT.** In this paper we examine the connection between unambiguity of cellular systems and the existence of Garden of Eden configurations in cellular automata. The examination includes both finite and infinite configurations. The connections are found by examining various properties of the global transition function.

**Introduction.** In 1962 Moore [5] showed that the existence of mutually erasable configurations in a two dimensional automata implies the existence of Garden of Eden configurations. In 1963 Myhill [6] showed that the existence of mutually indistinguishable configurations is necessary for the existence of Garden of Eden configurations. Amoroso and Cooper [1] have shown that Myhill's theorem does not apply to finite configurations but claim, without proof, that both Moore's and Myhill's theorems apply to infinite configurations. Arbib has redefined the concept of Garden of Eden in [2] and has shown that the existence of mutually indistinguishable configurations is both necessary and sufficient for the existence of Garden of Eden configurations. This theorem is not true.

The confusion in the field may be due to different definitions and concepts and prompts the systematic examination of the relation between different characteristics of the global transition function given in this paper.

**Definitions and notations.** A cellular automaton  $A$  is a system  $(I^n, g, Q, q_0, \sigma)$  where  
 $I^n$  is the underlying space and  $\alpha = (a_1, a_2, \dots, a_n)$  is a cell.  
 $g: I^n \rightarrow (I^n)^m$  is the *neighbourhood function* defined by  $g(\alpha) = (\alpha + \delta_1, \alpha + \delta_2, \dots, \alpha + \delta_m)$  where  $\delta_i (i = 1, 2, \dots, m) \in I^n$  is fixed.  
 $Q$  is a finite set of *states*.  
 $q_0 \in Q$  is a special state called the *quiescent state*.  
 $\sigma: Q^m \rightarrow Q$  is the *local transition function*, subject to the restriction  $\sigma(q_0^m) = q_0$ . This restriction assures us that no infinite configuration is successor for a finite configuration.

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Received by the editors April 3, 1972 and, in revised form, March 12, 1974.

AMS (MOS) subject classifications (1970). Primary 68A25, 94A30.

Key words and phrases. Garden of Eden, automata, tessellation automata, cellular automata.

A *configuration* in  $A$  is a mapping  $c: I^n \rightarrow Q$ . Let  $\mathcal{C}$  be the set of all configurations in  $A$ . A *finite configuration* in  $A$  is a configuration  $c$ , where the support is finite where  $\text{supp}(c) = \{\alpha \in I^n \mid c(\alpha) \neq q_0\}$ . Let  $\mathcal{C}_f$  be the set of all finite configurations in  $A$ .

The *neighbourhood state function*  $h: \mathcal{C} \times I^n \rightarrow Q^m$  is defined by:  
 $\forall c \in \mathcal{C}, \forall \alpha \in I^n: h(c, \alpha) = (c(\alpha + \delta_1), c(\alpha + \delta_2), \dots, c(\alpha + \delta_m))$ .

The *global transition function*  $\tau: \mathcal{C} \rightarrow \mathcal{C}$  is defined by:  $\forall c \in \mathcal{C}, \forall \alpha \in I^n: \tau(c)(\alpha) = \sigma(h(c, \alpha))$ . Let  $\tau_f = \tau|_{\mathcal{C}_f}$  be the restriction of  $\tau$  to  $\mathcal{C}_f$  ( $\tau_f: \mathcal{C}_f \rightarrow \mathcal{C}_f$ ).  $\hat{c}$  is a *subconfiguration* of  $c$  ( $\hat{c} < c$ ) iff  $c|_{\text{supp}(\hat{c})} = \hat{c}|_{\text{supp}(\hat{c})}$ .

The elements of  $G = \mathcal{C}_f \setminus \tau_f(\mathcal{C}_f) = \mathcal{C}_f \setminus \tau(\mathcal{C}_f)$  are called *Garden of Eden configurations*. This is *not* equivalent to the original definition of Moore.

$c \in \mathcal{C}_f$  is *strong Garden of Eden*<sup>1</sup> iff  $\exists S \subset I^n \ni$ :

- (1)  $S$  finite,
- (2)  $\text{supp}(c) \subset S$ ,
- (3)  $\forall \hat{c} \in \mathcal{C}_f: c^* \in G$  where

$$c^*(\alpha) = \begin{cases} c(\alpha), & \alpha \in S, \\ \hat{c}(\alpha), & \alpha \notin S. \end{cases}$$

Let  $G_s$  be the set of all strong Garden of Eden configurations.  $G_s = \emptyset$  iff there are no Garden of Eden configurations of the type defined by Moore. Arbib defines a configuration  $c$  as Garden of Eden when it satisfies:  $\forall \hat{c} \in \mathcal{C}_f: c < \hat{c} \Rightarrow \hat{c} \in G$ .

The set  $G_a$ , which is the set of configurations satisfying Arbib's definition, is included in  $G$ .

The global transition function on finite and infinite configurations. The close connection between properties of configurations and those of the global transition function is shown by the following facts.

- (1)  $\tau_f$  surjective  $\Leftrightarrow$  there are no Garden of Eden configurations ( $G = \emptyset$ ).
- (2)  $\tau_f$  injective  $\Leftrightarrow$  there are no mutually erasable configurations (see [5])  
 $\Leftrightarrow$  there are no mutually indistinguishable configurations (see [6]).

In the papers of Moore and Myhill, the proofs depended on the following inequality which was proved for the case  $n = 2$  and  $p = 2$ . To generalize their proofs to more dimensions and arbitrary finite neighbourhoods, we now give a general proof of the inequality:

**Lemma.**  $\forall A > 1, \forall r > 1, \forall n \geq 1, \forall p \geq 1, \exists k > 0: (A^{r^n} - 1)^{k^n} < A^{(kr-p)n}$ .

<sup>1</sup> A strong GOE configuration is equivalent to a GOE-restriction with respect to  $\mathcal{C}_f$  in the notation of Amoroso, Cooper and Patt.

**Proof.** If  $kr > p$ ,  $1 < i \leq n$ , then

$$(kr)^{n-i} p^i < (kr)^{n-1} \cdot p \wedge \binom{n}{i} \leq \binom{n}{[n/2]}$$

which means that

$$-(kr)^{n-1} \cdot p \cdot \binom{n}{[n/2]} \cdot n < \sum_{i=1}^n (-1)^i \binom{n}{i} (kr)^{n-i} \cdot p^i.$$

Let

$$K = r^{n-1} \cdot p \cdot n \cdot \binom{n}{[n/2]}.$$

If  $kr > p$  we then have

$$-Kk^{n-1} = -(kr)^{n-1} \cdot p \cdot n \binom{n}{[n/2]} < \sum_{i=1}^n (-1)^i \binom{n}{i} (kr)^{n-i} p^i = (kr - p)^n - (kr)^n.$$

Choose  $k > \max(p/r, K/\log_A(A^{r^n}/(A^{r^n} - 1)))$ . Then we have

$$\log_A((A^{r^n} - 1)/A^{r^n}) < -K/k \Rightarrow A^{r^n} - 1 < A^{r^n - K/k}$$

$$\Rightarrow (A^{r^n} - 1)^{k^n} < A^{(kr)^n - Kk^{n-1}} < A^{(kr-p)^n}. \quad \square$$

**Theorem 1.**  $G_S = \emptyset \Leftrightarrow \tau_f$  injective.

The proof is equivalent to the proofs in [5] and [6].

**Theorem 2.**  $\mathcal{C}_f \subset \tau(\mathcal{C}) \Rightarrow \tau_f$  injective.

**Proof.** Moore's argument from 1962.

**Theorem 3.**  $\tau_f$  injective  $\Rightarrow G_a = \emptyset$ .

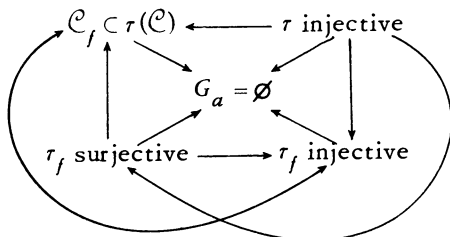
**Proof.** Myhill's argument from 1963.

**Theorem 4.**  $\mathcal{C}_f \setminus \tau(\mathcal{C}) = \emptyset \Leftrightarrow \tau_f$  injective.

**Theorem 5.**  $\tau$  injective  $\Rightarrow \tau_f$  surjective.

The proofs of Theorems 4 and 5 can be found in [7].

Since  $\tau_f$  surjective  $\Rightarrow \mathcal{C}_f \subset \tau(\mathcal{C})$  and  $\tau$  injective  $\Rightarrow \tau_f$  injective, closing gives the following diagram:



We now give 2 examples which demonstrate that several of the remaining implications are not true.

**Example 1.**

$$\{I, g(i) = (i, i + 1), \{0, 1\}, 0, \{00 \rightarrow 0, 01 \rightarrow 1, 10 \rightarrow 1, 11 \rightarrow 0\}\}.$$

Let  $L(c) = \min_i \{i | c(i) = 1\}$  and  $R(c) = \max_i \{i | c(i) = 1\}$ .  $\chi_A$  (where  $A \subset \mathbb{I}$ ) is the characteristic function defined by

$$\chi_A(i) = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}$$

$\tau$  is not injective because  $\chi_\emptyset$  is the successor of both  $\chi_I$  and itself. On the other hand  $\tau_f$  is injective because the following procedure uniquely determines the finite predecessor of  $c \in \mathcal{C}_f$  if it exists:

- (1)  $\hat{c}(i) = 0$  for  $i > R(c)$ ;
- (2) for  $i := R(c)$  down to  $L(c)$ :  $\hat{c}(i) = 1$  if  $\hat{c}(i + 1) = 1$  then  $1 - c(i)$  else  $c(i)$ ;
- (3)  $\hat{c}(i) = 0$  for  $i < L(c) + 1$ .

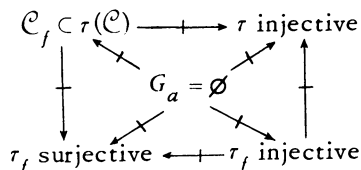
If we in Step 2 replace  $L(c)$  with  $-\infty$  and drop 3, we have a procedure for determination of an infinite predecessor. E.g.  $\mathcal{C}_f \subset \tau(\mathcal{C})$ . However  $\tau_f$  is not surjective because  $\chi_{\{j\}} = \tau_f(c) \Rightarrow L(c) = j + 1 \wedge R(c) = j$  which is absurd. Furthermore,  $G_a = \emptyset$  because  $\forall a, b : \tau_f(\chi_{\{2i | i \in [a, b]\}}) = \chi_{[2a-1, 2b]}$ .

**Example 2.**

$$\{I, g(i) = (i, i + 1), \{0, 1\}, 0, \{00 \rightarrow 0, 01 \rightarrow 1, 10 \rightarrow 1, 11 \rightarrow 1\}\}.$$

$\mathcal{C}_f \not\subset \tau(\mathcal{C})$  because  $\chi_{\{j\}} = \tau_f(c) \Rightarrow c(j) = 1 \Rightarrow \chi_{\{j\}}(j - 1) = 1$  which is absurd.  $G_a = \emptyset$  because  $a \leq b : \tau_f(\chi_{[a, b]}) = \chi_{[a-1, b]}$ . Finally  $\tau_f$  is not injective because  $\tau_f(\chi_{\{1, 3\}}) = \tau_f(\chi_{\{1, 2, 3\}}) = \chi_{\{0, 1, 2, 3\}}$ .

We have the following diagram:



From this it appears among other things that the definition of Garden of Eden appearing in this paper (and in many others) is too weak for the Garden of Eden theorem to have a converse, and Arbib's definition is too strong for the theorem to be true.

*Open problem.* Is it true that  $\tau_f$  surjective  $\Rightarrow \tau$  injective?

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