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A 2-PARAMETER CHEBYSHEV SET WHICH IS NOT A SUN

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ABSTRACT. Consider approximation with respect to the Chebyshev norm $\|g\| = \sup\{|g(x)| : 0 \le x \le 1\}$ on $[0, 1]$. A subset G of $C[0, 1]$ such that each $f \in C[0, 1]$ has a unique best approximation from G is called a *Chebyshev set*. It has been shown by the author that there exist Chebyshev sets which are not suns [2], but the examples given were essentially one-dimensional. An example is now given which is two-dimensional.

Let G be the set of functions of the form

$$\begin{aligned} F(A, x) &= (1 + a_1)\exp(-x/a_2), \qquad a_1 > 0,\ 0 < a_2 \le a_1, \\ &= 0 \qquad\qquad\qquad\qquad\qquad a_1 = 0 \text{ or } a_2 = 0. \end{aligned}$$

We claim that G is a Chebyshev set in $C[0, 1]$ which is not a sun.

First we show that best approximations exist to all $f \in C[0, 1]$. Let $\|F(A^k, \cdot)\| < M$. Since $F(A, 0) = 1 + a_1$, $\{a_1^k\}$ is bounded, hence $\{a_2^k\}$ is bounded. Assume without loss of generality that $\{a_1^k\} \to a_1$, $\{a_2^k\} \to a_2$. If a_1 or $a_2 = 0$, $F(A^k, \cdot) \to 0$ pointwise on $(0, 1]$. If $a_1, a_2 > 0$, $\{F(A^k, \cdot)\}$ converges uniformly to $F(A, \cdot)$. Hence G is *dense compact* in the terminology of [1] and best approximations exist to all $f \in C[0, 1]$.

Assertion. $a_1 < b_1$ and $a_2 < b_2$ imply that $F(A, \cdot) < F(B, \cdot)$.

Next we show uniqueness of best approximations. Given any nonzero approximant $F(B, \cdot)$, there exists by the assertion an approximation $F(A, \cdot)$ strictly between $F(B, \cdot)$ and zero. It follows that 0 is uniquely best when it is best. Suppose we have two distinct best approximations $F(A, \cdot)$ and $F(B, \cdot)$ to f. Suppose $a_2 = a_1$ and $b_2 = b_1$. Let c_1 be between a_1 and b_1 and $c_2 = c_1$. Then $F(C, \cdot)$ would be strictly between $F(A, \cdot)$ and $F(B, \cdot)$ by the assertion. $F(C, \cdot)$ would be a better approximation to f, giving a contradiction. Assume, therefore, that $a_2 < a_1$. Now there exist real numbers c_1, c_2 such that $F(A, x) = c_1 \exp(c_2 x)$. Moreover, there is a neighbourhood N of (c_1, c_2) such that for $(d_1, d_2) \in N$, $d_1 \exp(d_2 x)$ is in G. From the theory of Meinardus and Schwedt [3, Theorem 91], $f - F(A, \cdot)$ alternates 2 times. But $f - F(A, \cdot)$ alternating 2 times means that $F(A, \cdot)$ is uniquely

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best in approximations of the form $c_1 \exp(c_2 x)$ and so is uniquely best in G.

G is not a sun, as 0 is an isolated element.

Similar arguments show that the set of functions of the form

$$\begin{aligned} F(A, x) &= (1 + a_1)/(1 + x/a_2), \qquad && a_1 > 0,\ 0 < a_2 \leq a_1, \\ &= 0, && a_1 = 0 \text{ or } a_2 = 0, \end{aligned}$$

is a Chebyshev set which is not a sun. From these Chebyshev sets which are not suns, we can construct more complicated Chebyshev sets which are not suns. For the technique see [2, Theorem 2].

REFERENCES

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