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A CHARACTERIZATION OF PRIMITIVE BASES

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ABSTRACT. The concept of primitive base is characterized in terms of primitive sequences. A characterization of essentially T_1 spaces having bases of countable order is a corollary.

1. **Introduction.** The concept of primitive base is fundamentally related to spaces having bases of countable order (see Theorem 2.6). It therefore underlies in a basic way such important topological structures as developable spaces and metrizable spaces. The main purpose of this article is to characterize primitive base in terms of primitive sequences, i.e., by a certain structure defined with the use of a sequence of well-ordered open coverings. This result shows that spaces having primitive bases are prototypes of both spaces having θ-bases (equivalently, quasi-developable spaces [2]) and of spaces having bases of countable order. The presence in such spaces of what we call a primitive sequence of basic type (see 4.2) permits the development of an extensive and harmonious theory [8]–[10], [12]–[14] wider in scope than the theories of quasi-developable spaces or of spaces having bases of countable order.

In §2 we make some detailed comparisons. In §3 we prove the main theorem. In §4 we obtain a new characterization of spaces having bases of countable order. In §5 we discuss completeness. We shall discuss elsewhere additional properties of spaces having primitive bases and a natural generalization to non-first-countable spaces arising from the main theorem.

Notation and terminology. Sequences such as $\langle H_n: n \in N\rangle$ will be denoted by H. Recall that a space X is *essentially* T_1 [12] if and only if for all $x, y \in X$, $x \in \overline{\{y\}}$ implies $y \in \overline{\{x\}}$ (see [3] also). The letter N denotes the set of positive integers. The notations $<$ and $\leq$ are used ambiguously to denote orderings whose domains and ranges are contextually clear.

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2. **Primitive base.** We define *primitive base* and show that it generalizes base of countable order and θ-base. Theorem 2.6, stated without proof, gives a precise connection with essentially T_1 spaces having bases of countable order (see also 4.3).

2.1. **Definition** [11]. A topological space X is said to have a *primitive base* if and only if there exists a sequence $\mathfrak{W}$ whose terms are well-ordered collections of open sets such that, for all $x \in X$, if U is open and $x \in U$, then there exist integers k and n such that x is in n elements of $\mathfrak{W}_k$ and the nth such element is a subset of U.

2.2. **Definition** [12]. A topological space is said to have a *θ-base* if and only if there exists a collection $\mathfrak{B}$ of open sets in X such that $\mathfrak{B} = \bigcup_{n \in N} \mathfrak{B}_n$ and for each $x \in X$, if U is open and $x \in U$, then there exists $n \in N$ such that $\{B \in \mathfrak{B}_n : x \in B\}$ is finite and some member of this set is a subset of U.

2.3. **Theorem** [2]. *A topological space X has a θ-base if and only if it is quasi-developable, i.e., there exists a sequence $\mathcal{G}$ of collections of open subsets of X such that for each $x \in X$ if U is open and $x \in U$ there exists n such that $x \in \mathrm{st}(x, \mathcal{G}_n) \subseteq U$.*

2.4. **Theorem.** *Every topological space which has a θ-base, or is quasi-developable, or is an essentially T_1 space having a base of countable order, has a primitive base.*

Proof. The first two statements are clear from the definitions and 2.3. The last statement follows from 3.6 and 4.1.

2.5. **Example.** Let X denote the Michael line [4] and ω_1 the space of countable ordinals with the order topology. Then the topological sum $X \oplus \omega_1$ and the topological product $X \times \omega_1$ are spaces which have primitive bases but which have neither a base of countable order nor a θ-base.

Proof. The space X has a base consisting of the irrational singletons and the collection of all balls $B(x, 1/n)$ where x is rational and $n \in N$. Let $\{x_n : n \in N\}$ be the range of a bijection between N and the set of rational numbers. Let $\mathfrak{B}_{00} = \{\{y\} : y \text{ is irrational}\}$ and $\mathfrak{B}_{nm} = \{B(x_n, 1/m)\}$ for $n, m \in N$. The collection $\mathfrak{B}_{00} \cup \bigcup_{n,m \in N} \mathfrak{B}_{nm}$ is a θ-base for X. The space ω_1 has a base of countable order but is not developable [15 (in different terminology)]. That ω_1 does not have a θ-base follows from the result of [2] that a linearly ordered space is quasi-developable if and only if it is paracompact, since ω_1 is not paracompact. The property of having a θ-base is hereditary as is that of having a base of countable order [12]. Hence neither $X \oplus \omega_1$ nor $X \times \omega_1$ can have a θ-base or a base of countable order. That $X \oplus \omega_1$ has a primitive base is clear from 2.4 and the definition of topological sum. That $X \times \omega_1$ has a primitive base follows from 2.4 and 3.9.

The following theorem, whose proof is made accessible by Theorem 3.6, will be proved elsewhere. For *set of interior condensation* see [5].

2.6. **Theorem** [11], [13]. *A topological space is essentially T_1 and has a base of countable order if and only if it has a primitive base and closed sets are sets of interior condensation locally.*

This result makes interesting contrast to the next two (known) theorems.

2.7. **Theorem** [12]. *A topological space is developable if and only if it has a θ-base and closed sets are G_δ sets.*

A theorem equivalent in content to 2.7, in view of Theorem 2.3, is the next one, discovered by Bennett independently of 2.3 and 2.7.

2.8. **Theorem** [1]. *A topological space is developable if and only if it is quasi-developable and closed sets are G_δ sets.*

3. **Primitive sequences and primitive bases.** We first present some terminology of the theory of primitive sequences [6], and then characterize essentially T_1 spaces having primitive bases.

3.1. **Definition.** Let $(\mathcal{Z}, \leq)$ be a well-ordered collection of sets. For each $W \in \mathcal{Z}$ let $p(W, \mathcal{Z})$ denote $\{x \in W\colon W' \in \mathcal{Z}$ and $W' < W$ implies $x \notin W'\}$. The set $p(W, \mathcal{Z})$ is called the *primitive part of W (in $\mathcal{Z}$)*.

In [6], primitive sequences are defined in a general way. Here, to simplify the exposition, we define only *open primitive sequences.*

3.2. **Definition.** Let X be a topological space. An *open primitive sequence of X* is a sequence $\mathcal{H}$ of well-ordered open covers of X such that, for each $n \in N$:

(a) For all $H \in \mathcal{H}_n$, $p(H, \mathcal{H}_n) \neq \emptyset$.

(b) If $j < n$ and $p(H, \mathcal{H}_n) \cap p(H', \mathcal{H}_j) \neq \emptyset$, then $H \subseteq H'$.

3.3. **Definition.** Let $\mathcal{H}$ be an open primitive sequence of a space X. A *primitive representative of $\mathcal{H}$* is a sequence H such that for all $n \in N$, $p(H_n, \mathcal{H}_n) \cap p(H_{n+1}, \mathcal{H}_{n+1}) \neq \emptyset$.

3.4. *Notation.* If $\mathcal{H}$ is a primitive sequence, $PR(\mathcal{H})$ denotes $\{H\colon H$ is a primitive representative of $\mathcal{H}\}$.

3.5. **Definition.** If $\mathcal{H}$ is a primitive sequence, then for all $H \in PR(\mathcal{H})$, $pc(H)$ denotes $\bigcap_{n \in N} p(H_n, \mathcal{H}_n)$. This set is called the *primitive core* of H.

3.6. **Main theorem.** *A topological space is essentially T_1 and has a* *primitive base if and only if it has an open primitive sequence* $\mathfrak{H}$ *such that for all* $H \in PR(\mathfrak{H})$, *if* $pc(H) \neq \emptyset$, *then* $\{H_n : n \in N\}$ *is a base at each element of* $\bigcap_{n \in N} H_n$.

Proof. *Necessity.* Let $\mathfrak{W}$ be a sequence of well-ordered collections related to a space X as in 2.1. Assume X is essentially T_1. Call $(k, n) \in N \times N$ *admissible for* $x \in X$ if x is in n elements of $\mathfrak{W}_k$. Let $A(k, n) = \{x \in X: (k, n)$ is admissible for $x\}$. For $x \in A(k, n)$ let $W(x, i)$ denote the ith element of $\mathfrak{W}_k$ that contains x, let $\mathfrak{W}(x, n) = \{W(x, i): i \leq n\}$, and let $V(x) = \bigcap\{W(x, i): i \leq n\}$. Let $\mathfrak{V}(k, n) = \{V(x): x \in A(k, n)\}$. Then for $x, y \in A(k, n)$, $V(x) = V(y)$ if and only if $\mathfrak{W}(x, n) = \mathfrak{W}(y, n)$. Let $<$ be defined on $\mathfrak{V}(k, n)$ by $V(x) < V(y)$ if and only if $V(x) \neq V(y)$ and if $m = \min\{i: W(x, i) \neq W(y, i)\}$, then $W(x, m)$ precedes $W(y, m)$ in $\mathfrak{W}_k$. It is readily verified that $\leq$ well orders $\mathfrak{V}(k, n)$. Furthermore $x \in p(V(x), \mathfrak{V}(k, n))$ for all $x \in A(k, n)$. If $\mathfrak{V}(k, n)$ covers X let $\mathfrak{V}'(k, n) = \mathfrak{V}(k, n)$. If not, let $\mathfrak{V}'(k, n) = \mathfrak{V}(k, n) \cup \{X\}$ with an ordering which extends that of $\mathfrak{V}(k, n)$ and makes X follow all other elements. If $A(k, n) = \emptyset$, let $\mathfrak{V}'(k, n) = \{X\}$. Let $\mathfrak{U}_{(k+n-1)(k+n-2)/2+k}$ denote $\mathfrak{V}'(k, n)$ for all $(k, n) \in N \times N$. Let $\mathfrak{H}_1 = \mathfrak{U}_1$. Suppose $\mathfrak{H}_1, \cdots, \mathfrak{H}_n$ have been defined and satisfy the conditions of 3.2 for $1 \leq i \leq n$. Call a set W of $(n+1)$-form if there exist $U \in \mathfrak{U}_{n+1}$ and $H \in \mathfrak{H}_n$ such that $p(U, \mathfrak{U}_{n+1}) \cap p(H, \mathfrak{H}_n) \neq \emptyset$ and $W = U \cap H$. If W is of $(n+1)$-form, then its representation is unique; i.e., if $U, U' \in \mathfrak{U}_{n+1}$ and $H, H' \in \mathfrak{H}_n$, and

$$p(U, \mathfrak{U}_{n+1}) \cap p(H, \mathfrak{H}_n) \neq \emptyset \neq p(U', \mathfrak{U}_{n+1}) \cap p(H', \mathfrak{H}_n),$$

and $U \cap H = U' \cap H'$, then $U = U'$ and $H = H'$. In what follows, if W is written in $(n+1)$-form $U \cap H$, then $U \in \mathfrak{U}_{n+1}$ and $H \in \mathfrak{H}_n$. Let $\mathfrak{H}_{n+1} = \{W: W$ is of $(n+1)$-form$\}$ ordered by $W < W'$ if and only if $W \neq W'$ and if $U \cap H$ and $U' \cap H'$ are the respective representations of W and W', then either (a) U precedes U' in $\mathfrak{U}_{n+1}$, or (b) $U = U'$ and H precedes H' in $\mathfrak{H}_n$. It may be verified that $\leq$ is a well ordering of $\mathfrak{H}_{n+1}$. Furthermore, if $W \in \mathfrak{H}_{n+1}$ and $W = U \cap H$, then

$$p(W, \mathfrak{H}_{n+1}) = p(U, \mathfrak{U}_{n+1}) \cap p(H, \mathfrak{H}_n).$$

For if $x \in p(W, \mathfrak{H}_{n+1})$, then $x \in U \cap H$. If $x \in p(U', \mathfrak{U}_{n+1}) \cap p(H', \mathfrak{H}_n)$, then $W' = U' \cap H' \in \mathfrak{H}_{n+1}$. Since $U' \leq U$ and $H' \leq H$ it follows that $W' \leq W$. Thus $W = W'$. These considerations establish the equality. Clearly $\mathfrak{H}_{n+1}$ is an open cover of X since $\mathfrak{H}_n$ is, and $p(H, \mathfrak{H}_{n+1}) \neq \emptyset$ for all $H \in \mathfrak{H}_{n+1}$. Suppose $x \in p(W, \mathfrak{H}_{n+1}) \cap p(W', \mathfrak{H}_n)$ and $W = U \cap H$. Then $x \in p(U, \mathfrak{U}_{n+1}) \cap p(H, \mathfrak{H}_n)$.

Hence $H = W'$. Thus $W \subseteq W'$. Condition (b) of 3.3 follows from this. Thus an open primitive sequence $\mathcal{H}$ of X may be defined by induction.

Suppose $H \in PR(\mathcal{H})$ and $x \in pc(H)$. Let D be open and $x \in D$. Then there exist k and n such that x is in n elements of W_k and the nth such element is a subset of D. Therefore $V(x) \in \mathcal{O}(k, n)$ and $V(x) \subseteq D$. Since $x \in p(V(x), \mathcal{O}(k, n))$ it follows for $j = (k + n - 2)(k + n - 1)/2 + k$ that $x \in p(V(x), \mathcal{U}_j)$. If $j = 1$, $V(x) = H_1 \in \mathcal{H}_1$ and $V(x) \subseteq D$. If $j > 1$, then there exists $H' \in \mathcal{H}_{j-1}$ such that $x \in V(x) \cap H' \in \mathcal{H}_j$, and $V(x) \cap H' \subseteq D$. But $x \in p(H_j, \mathcal{H}_j)$ implies $H_j = V(x) \cap H'$. Thus $\{H_n: n \in N\}$ is a base at each point of $pc(H)$.

Suppose $z \in \{\overline{y}\} \cap pc(H)$. Then $y \in \{\overline{z}\}$ by essential T_1-ness. If $y \in p(W, \mathcal{H}_n)$, then $z \in W$, so that $H_n \leq W$. Since $y \in H_n$, $W = H_n$. Thus $\{\overline{y}\} \cap pc(H) \neq \emptyset$ implies $\{\overline{y}\} \subseteq pc(H)$. Suppose $z \notin pc(H)$. Then $pc(H) \subseteq X \backslash \{\overline{z}\}$. Thus $z \notin H_j$ for some $j \in N$. Hence $pc(H) = \bigcap_{n \in N} H_n$ and the proof is complete.

Sufficiency. Suppose such a sequence $\mathcal{H}$ exists in a space X. Then for all $x \in X$, there is $H \in PR(\mathcal{H})$ such that $x \in pc(H)$. If D is open and $x \in D$, then there is a k such that $H_k \subseteq D$. Thus $(k, 1)$ is an admissible pair for x. Therefore X has a primitive base. Suppose $x, y \in X$ and $y \in \{\overline{x}\}$ and $x \in pc(H)$. If $y \in pc(W)$, then $x \in \bigcap\{W_n: n \in N\}$. Thus for each n there is $j > n$ such that $W_j \subseteq H_n \cap W_n$. Hence $y \in \bigcap\{H_n: n \in N$ so that $x \in \{\overline{y}\}$. Thus X is essentially T_1.

3.7. **Remark.** Note that the axiom of choice is not used to prove 3.6.

3.8. **Example.** The Sierpiński space $S = \{\emptyset, 1\}$ with topology $\{\emptyset, \{\emptyset\}, S\}$ has a primitive base but is not essentially T_1. It has exactly two primitive sequences: one has each term equal to $\{S\}$; the other is defined by $\mathcal{H}_n = \{\{\emptyset\}, S\}$ where $\{\emptyset\} < S$. The only primitive representatives of $\mathcal{H}$ are either those whose terms are eventually $\{\emptyset\}$ or the primitive representative H all of whose terms are S. Note that $pc(H) = \{1\}$, but H is not a base at each point of $\bigcap_{n \in N} H_n = S$.

From 3.6 and the calculus of primitive sequences we obtain the next two theorems. Proofs will be submitted elsewhere.

3.9. **Theorem.** *The concept of primitive base is hereditary and countably productive.*

3.10. **Theorem.** *If an essentially T_1 topological space X is the union of open sets each having a primitive base in the relative topology, then X has a primitive base.*

4. Spaces having bases of countable order. Theorem 4.1, whose proof may be obtained from those of Theorems 1 and 2 of [12], compares instructively with 3.6 and proves part of 2.4. Theorem 4.3 puts in sharp focus the distinction between 3.6 and 4.1.

4.1. **Theorem** [12]. *A topological space is essentially T_1 and has a base of countable order if and only if it has a primitive sequence $\mathcal{H}$ such that, for all $H \in PR(\mathcal{H})$, if $\bigcap_{n \in N} H_n \neq \emptyset$, then $\{H_n : n \in N\}$ is a base at each point of $\bigcap_{n \in N} H_n$.*

4.2. **Definition.** Suppose X is a topological space and $\mathcal{H}$ is an open primitive sequence of X satisfying the conditions of (3.6). Then $\mathcal{H}$ will be called a *primitive sequence (of X) of basic type.*

4.3. **Theorem.** *A topological space is essentially T_1 and has a base of countable order if and only if it has a primitive sequence $\mathcal{H}$ of basic type such that for all $H \in PR(\mathcal{H})$, if $\bigcap_{n \in N} H_n \neq \emptyset$, then $pc(H) \neq \emptyset$.*

Proof. *Necessity.* Suppose X is essentially T_1 and has a base of countable order. Let $\mathcal{H}$ be a sequence relative to X as in 4.1, and let $H \in PR(\mathcal{H})$ and $x \in \bigcap_{n \in N} H_n$. There exists $W \in PR(\mathcal{H})$ such that $x \in pc(W)$. Thus each $W_n \leq H_n$. Since $\{H_n : n \in N\}$ is a base at x, there exists $j > n$ such that $H_j \subseteq H_n \cap W_n$. Since $p(H_j, \mathcal{H}_j) \subseteq p(H_n, \mathcal{H}_n)$, it follows that $H_n \leq W_n$. Thus $pc(H) = \bigcap_{n \in N} H_n$.

4.4. **Remark.** This makes precise a sense in which essentially T_1 spaces having bases of countable order are *complete spaces* in the class of spaces having primitive bases.

5. Completeness. A natural question arises concerning completeness. We discuss it for the regular T_0 case although an analogous discussion can be given for pararegular [7] spaces. Recall that a regular T_0 space having a base of countable order is called an *Aronszajn space* [6] and a monotonically complete member of this class is called a *complete Aronszajn space.* Does there exist for the class of regular T_0 spaces having a primitive sequence of basic type a "natural" class of complete spaces? Theorem 5.3 shows that a natural method of defining a complete space in this case (one analogous to that used for Aronszajn spaces) yields the complete Aronszajn spaces rather than a new class. Theorems 5.1 and 5.2 imply 5.3.

5.1. **Theorem.** *Suppose X is a topological space having a primitive sequence $\mathcal{H}$ of basic type. Then the following are equivalent:*

(a) *$H \in PR(\mathcal{H})$ implies $\bigcap_{n \in N} H_n \neq \emptyset$ and $\{H_n : n \in N\}$ is a base at each point of $\bigcap_{n \in N} H_n$.*

(b) *$H \in PR(\mathcal{H})$ implies $pc(H) \neq \emptyset$.*

Proof. Clearly (b) implies (a). The proof of 4.3 shows that (a) implies (b).

5.2. **Theorem.** *A topological space is a complete Aronszajn space if and only if it is regular T_0 and has an open primitive sequence $\mathcal{H}$ such that for all $H \in PR(\mathcal{H})$, $\bigcap_{n \in N} H_n \neq \emptyset$ and $\{H_n : n \in N\}$ is a base at each point of $\bigcap_{n \in N} H_n$.*

Proof. A proof for this may be obtained from the proofs of Theorems 3.2, 3.3, and Lemma 2.4 of [6].

5.3. **Theorem.** *A topological space is a complete Aronszajn space if and only if it is a regular T_0 space having a primitive sequence $\mathcal{H}$ of basic type such that for all $H \in PR(\mathcal{H})$, $pc(H) \neq \emptyset$.*

Added in proof. (1) A proof of Theorem 2.6 is to appear in a paper by the authors in the Canadian Journal of Mathematics in 1975. (2) The proof of Theorem 3.6 shows that *essentially* T_1 may be omitted in its statement if $\bigcap_{n \in N} H_n$ is replaced by $pc(H)$. (Dr. T. M. Phillips proved a similar theorem in his 1974 University of Oklahoma dissertation.)

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