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A PERTURBATION THEOREM FOR PARTIAL DIFFERENTIAL OPERATORS

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ABSTRACT. Existence and uniqueness of solutions to a class of operator equations is shown. This class includes a large subclass of partial differential operators.

1. **Introduction.** By a *closed bordered chain of orthoprojectors* [1], [3], [5] on a Hilbert space H we mean a set β of orthogonal projections on H which are linearly ordered (i.e., $P_1 < P_2$ if $P_1(H) \subset P_2(H)$), contain the zero and identity operators, and is closed in the sense that whenever P_i, $i = 1, 2, \cdots$, is a sequence of projections in β such that $\lim_{i\to\infty} P_i(x) = Px$ exists for all x in H, then P is in β. By a partition of the chain β we mean a finite set of operators $z = \{P_0, P_1, \cdots, P_n\}$, such that $0 = P_0 < P_1 < \cdots < P_n = I$.

Using the above structure one may define the *diagonal integral* [5], [6] of an operator T on H by

$$\int dP\, T\, dP = \lim_{z \in \mathfrak{z}} \sum_{i=1}^{n} [P_i - P_{i-1}]\, T [P_i - P_{i-1}], \tag{1.1}$$

whenever the limit exists. Here the limit is taken in the uniform operator topology over the directed set of all partitions $\mathfrak{z}$ of β.

We say that an operator is *diagonal* (relative to β) if $PT = TP$ for all P in β, *triangular* if $PT = PTP$ for all P in β, and *strictly triangular* if it is triangular and $\int dP\, T\, dP = 0$. The above terminology is motivated by the finite dimensional case, in which T may be represented by a (block) triangular matrix using a basis generated by β when it is triangular; and similarly for strictly triangular and diagonal operators. We note that in much of the litera-

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ture triangularity and strict triangularity are defined via appropriate "triangular integrals" [1], [5], [6] rather than as above. Such an approach, however, presupposes that T is bounded (for the integrals to converge uniformly), whereas no such restriction is required for the validity of the above definitions. Of course, in the bounded cases the two formulations are equivalent [3], [4], [6], [7].

Using the above notation with a fixed, though arbitrary, closed bordered chain β of orthogonal projections on a Hilbert space H we may state our main results. For this purpose we say that an operator valued function, $A(t)$, densely defined on H for each t in $[0, T]$ admits a resolvent if the equation

$$R_t(t, s) = A(t)R(t, s), \qquad R(s, s) = I, \tag{1.2}$$

admits a unique bounded operator valued solution continuous on $[0, T]$.

Theorem. *Let $A(t)$ be a family of diagonal operators densely defined on H for each t in $[0, T]$ which admits a resolvent. Then for any strictly triangular operator B the Cauchy problem for the equation*

$$X_t = [A(t) + B]X + f, \qquad X(0) = X_0,$$

has a unique solution in $L_2\{[0, T], H\}$.

Corollary. *If A is a triangular operator densely defined on H and $\int dP\, A\, dP$ exists, then the Cauchy problem for the equation*

$$X_t = AX + f, \qquad X(0) = X_0,$$

has a unique solution on some interval $[0, T]$.

If one lets Ω be a region in R^n and H be $L_2\{\Omega, R\}$, then a chain may be constructed from a linearly ordered (by containment) family of subsets $\mathcal{S}$ of Ω, by letting P_S be the projection defined by

$$(P_S f)(x) = \begin{cases} f(x), & x \in S, \\ 0, & x \notin S, \end{cases} \tag{1.3}$$

and letting $\beta = \{P_S : S \in \mathcal{S}\}$. Indeed, with this chain, every partial differential operator, being local, is diagonal. As such, any family of partial differential operators, $A(t)$, which admits a resolvent, satisfies the hypotheses of the Theorem and it is necessary only that one choose the family of subsets β so as to render the perturbing operator B strictly triangular.

The Theorem is a generalization of a recent result of De Santis [3], [4] on the existence of solutions to a generalized Volterra equation, which we extend in this paper to the unbounded case. The next section of the paper is devoted to the formulation of the appropriate lemmas on the solution of generalized Volterra equations, and the proof of the Theorem is given in the third section. Finally, §4 is devoted to an example.

2. Generalized Volterra equations.

Lemma 1. *Let L be a strictly triangular operator densely defined on H. Then $(I-L)^{-1}$ exists on a dense subspace of H and is triangular.*

Proof. Since L is strictly triangular, $\int dP\, L\, dP = 0$. Hence, there exists a partition $z = \{P_0, P_1, \cdots, P_n\}$ such that

$$\left\| \sum_{i=1}^{n} \Delta P_i\, L\, \Delta P_i \right\| < 1, \tag{2.1}$$

where $\Delta P_i = [P_i - P_{i-1}]$. Since $P_i > P_{i-1}$, ΔP_i is a projection and

$$\sum_{i=1}^{n} \Delta P_i = \sum_{i=1}^{n} [P_i - P_{i-1}] = P_n - P_0 = I. \tag{2.2}$$

If we let $H_i = P_i(H)$, we may decompose H into the direct sum of the H_i; i.e., $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$. Now, we may represent L as an $n \times n$ matrix of operators on the subspaces H_i via $(L)_{ij} = (\Delta P_i\, L\, \Delta P_j)$, where the term $(L)_{ij}$ is zero whenever $i < j$. This follows from the fact that L is triangular; i.e., for $i < j$,

$$\begin{aligned} \Delta P_i\, L\, \Delta P_j &= [P_i - P_{i-1}]\, L\, [P_j - P_{j-1}] = [P_i - P_{i-1}] P_i L [P_j - P_{j-1}] \\ &= [P_i - P_{i-1}] P_i L P_i [P_j - P_{j-1}] = [P_i - P_{i-1}] P_i L [P_i - P_i] = 0, \end{aligned}$$

since $P_i \le P_{j-1} < P_j$, and thus $P_i P_j = P_i P_{j-1} = P_i$.

Representing the identity operator in the same basis we may thus represent $(I - L)$ via the matrix

$$(I - L) = \begin{bmatrix} (I - \Delta P_1 L \Delta P_1) & 0 & 0 & \cdots & 0 \\ \Delta P_2 L \Delta P_1 & (I - \Delta P_2 L \Delta P_2) & 0 & \cdots & 0 \\ \Delta P_3 L \Delta P_1 & \Delta P_3 L \Delta P_2 & (I - \Delta P_3 L \Delta P_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \Delta P_n L \Delta P_1 & \Delta P_n L \Delta P_2 & \cdots & & (I - \Delta P_n L \Delta P_n) \end{bmatrix} \tag{2.3}$$

Finally, since the terms $\Delta P_i\, L\, \Delta P_i$ have orthogonal ranges, inequality (2.1) implies that $\|\Delta P_i\, L\, \Delta P_i\| < 1$ for each i. As such, the diagonal terms in the matrix representation of $(I - L)$ are all invertible, implying that the matrix itself is invertible by standard triangular matrix techniques, even though the off-diagonal entries in the matrix (and its inverse) may be unbounded. Q.E.D.

The lemma is a straightforward generalization of the linear version of the recent theorem of De Santis [3], [4], in which the boundedness condition on L has been dropped (which, in turn, generalizes the results of Brodskiĭ [2] and Gohberg and Kreĭn [1] for the cases in which L is compact and Hilbert-Schmidt respectively). A second lemma which we will need is also derived from a theorem of De Santis [6], [7] with an appropriate relaxation of the boundedness condition.

Lemma 2. *Let M be a bounded diagonal operator and L be strictly triangular. Then ML is strictly triangular.*

Proof. Since M is diagonal and L is triangular, $PML = MPL = MPLP = PMLP$, showing that ML is triangular. Now, for any partition $z = \{P_0, P_1, \dots, P_n\}$,

$$\begin{aligned}\left\|\sum_{i=1}^{n}[P_i - P_{i-1}]ML[P_i - P_{i-1}]\right\| &= \left\|\sum_{i=1}^{n} M[P_i - P_{i-1}]L[P_i - P_{i-1}]\right\| \\ &= \left\|M\sum[P_i - P_{i-1}]L[P_i - P_{i-1}]\right\| \qquad (2.4)\\ &\leq \|M\|\left\|\sum_{i=1}^{n}[P_i - P_{i-1}]L[P_i - P_{i-1}]\right\|,\end{aligned}$$

where the commutation of the P_i's with M is justified since M is diagonal. Since L is strictly triangular the summation on the right side of (2.4) can be made arbitrarily small for sufficiently fine partitions; since M is bounded, the summation on the left side of (2.4) can be made arbitrarily small also. Thus, $\int dP\, ML\, dP = 0$, and ML is strictly triangular, as required. Q.E.D.

Lemma 2 is, in fact, also true for M triangular (the proofs given in the literature [6], [7] for the bounded case generalize in the obvious manner), though we will not need that fact here. Indeed, this extension of the lemma leads to a perturbation theorem for the existence of solutions of generalized Volterra equations to the effect that $(I - (K + L))$ has a densely defined triangular inverse (which is bounded when L is bounded) if $(I - K)$ has a bounded triangular inverse and L is any strictly triangular perturbation.

3. **Proof of the Theorem.** Since $A(t)$ admits a resolvent R on $[0, T]$, the solution of the Cauchy problem

$$X_t = A(t)X + g, \qquad X(0) = X_0, \tag{3.1}$$

is given by

$$X(t) = R(t, 0)X_0 + \int_0^t R(t, s)g(s)\,ds, \qquad 0 \leq t \leq T. \tag{3.2}$$

Thus, if we let $g = f + BX$, we have for the perturbed Cauchy problem

$$X_t = [A(t) + B]X + f, \qquad X(0) = X_0, \tag{3.3}$$

that

$$X(t) = R(t, 0)X_0 + \int_0^t R(t, s)[B(s)X(s) + f(s)]\,ds, \qquad 0 \leq t \leq T, \tag{3.4}$$

or equivalently that

$$X(t) - \int_0^t R(t, s)BX(s)\,ds = R(t, 0)X_0 + \int_0^t R(t, s)f(s)\,ds = Y(t), \qquad 0 \leq t \leq T, \tag{3.5}$$

which is in the form of a generalized Volterra equation

$$X - KX = y \tag{3.6}$$

on the Hilbert space $\tilde{H} = L_2\{[0, T], H\}$. To prove the Theorem it therefore suffices to show that the operator K defined on $\tilde{H}$ by

$$(KX)(t) = \int_0^t R(t, s)BX(s)\,ds \tag{3.7}$$

satisfies the hypotheses of Lemma 2 for some chain $\tilde{\beta}$.

We begin by defining a chain $\tilde{\beta}$ on $\tilde{H}$ via

$$\tilde{\beta} = \{P: \tilde{H} \to \tilde{H} | (\tilde{P}X)(t) = PX(t),\ P \in \beta\}.$$

This is the natural lifting of the chain β on H to a chain on $\tilde{H}$ and it inherits its properties from β. As a preliminary to showing that K is strictly triangular (relative to $\tilde{\beta}$) we will show that $R(t, s)$ is diagonal (relative to β). Since

$$R_t = AR, \qquad R(s, s) = I \tag{3.8}$$

on the interval $[0, T]$, multiplying (3.8) by P on the left yields $(PR)_t = PAR = A(PR)$, $(PR)(s, s) = PI = P$, where we have used the fact that A is diagonal. Similarly, multiplying (3.8) on the right by P yields $(RP)_t = A(RP)$, $(RP)(s, s) = IP = P$. RP and PR are thus both solutions to the same differential equation with the same initial condition, and having assumed that equation (3.1) has a unique resolvent we are assured that $R(t, s)P = PR(t, s)$, $0 \leq t \leq T$ and $0 \leq s \leq T$; i.e., that $R(t, s)$ is diagonal for all s and t in $[0, T]$. If we apply Lemma 2, $R(t, s)B$ is thus assured to be strictly triangular (relative to β) for all s and t.

Finally, for any $\tilde{P}$ in $\tilde{\beta}$, we have

$$\tilde{P}KX = P\int_0^t R(t, s)BX(s)\,ds = \int_0^t PR(t, s)BX(s)\,ds$$

$$= \int_0^t PR(t, s)BPX(s)\,ds = P\int_0^t R(t, s)B(\tilde{P}X)(s)\,ds = \tilde{P}K\tilde{P}X,$$

showing that K is triangular (relative to $\tilde{\beta}$). Here we have used the fact that $R(t, s)B$ is triangular (relative to β) to justify the equality $PR(t, s)B = PR(t, s)BP$. For any s, $R(t, s)B$ is strictly triangular. Therefore $\int dP\,R(t, s)B\,dP = 0$, or equivalently

$$\left\| \sum_{i=1}^{n} [P_i - P_{i-1}]\, R(t, s)B\, [P_i - P_{i-1}] \right\| \to 0 \tag{3.9}$$

for sufficiently refined partitions, for each fixed s and t. However, since $R(t, s)$ is continuous with respect to each variable and defined over a compact set, it must also go to zero uniformly in s for each fixed t. As such, for any $\epsilon > 0$, there exists a partition $z = \{P_0, P_1, \cdots, P_n\}$ such that

$$\left\| \sum_{i=1}^{n} \Delta P_i\, R(t, s)B\, \Delta P_i \right\| < \epsilon, \qquad 0 \leq s, \quad t \leq T. \tag{3.10}$$

Letting $\tilde{z}$ be the partition of $\tilde{\beta}$ induced by z we thus have

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} [\tilde{P}_i - \tilde{P}_{i-1}]\, K [\tilde{P}_i - \tilde{P}_{i-1}] X \right\|_{\tilde{H}}^2 \\
&\quad = \int_0^T \left\| \sum_{i=1}^{n} [\tilde{P}_i - \tilde{P}_{i-1}]\, K [\tilde{P}_i - \tilde{P}_{i-1}] X(t) \right\|_{H}^2 dt \\
&\quad = \int_0^T \left\| \sum [P_i - P_{i-1}] \int_0^t R(t, s)B[P_i - P_{i-1}]X(s)\,ds \right\|_H^2 dt \\
&\quad = \int_0^T \left\| \int_0^t \sum_{i=1}^{n} [P_i - P_{i-1}]\, R(t, s)B[P_i - P_{i-1}]X(s)\,ds \right\|_H^2 dt \\
&\quad \leq \int_0^T \int_0^t \left\| \sum_{i=1}^{n} [P_i - P_{i-1}]\, R(t, s)B[P_i - P_{i-1}] \right\|_H^2 \|X(s)\|_H^2\, ds\, dt \\
&\quad \leq \int_0^T \int_0^t \epsilon^2 \|X(s)\|_H^2\, ds\, dt \leq \int_0^T \epsilon^2 \int_0^T \|X(s)\|_H^2\, ds\, dt = \epsilon^2 T \|X\|_{\tilde{H}}^2
\end{aligned} \tag{3.11}$$

which goes to zero for sufficiently fine partitions. Thus $\int dP\, K\, dP = 0$, verifying that K is strictly triangular (relative to $\tilde{\beta}$) and hence, via Lemma 1, that equation (3.5) has a unique solution, as required. The proof of the Corollary may be obtained by decomposing A as $A = \int dP\, A\, dP + B$. Now $\int dP\, A\, dP$ is diagonal and bounded (since the integral converges in the uniform topology). Hence, it admits a resolvent on some interval $[0, T]$, and B is strictly triangular (since the diagonal integral is idempotent). As such, the Corollary follows from the Theorem. Q.E.D.

We conjecture that the Theorem can be strengthened to include the case where A is merely triangular rather than diagonal. Indeed, since Lemma 1 is valid in the triangular case, to extend the Theorem it would suffice to show that the resolvent of a triangular operator A is always triangular. In fact, for triangular A it is readily verified that both PR and PRP are solutions of the equations $S_t = (PA)S$ with initial condition P; hence, it is only required that we verify that this equation has a unique solution.

4. **An example.** Consider an LC transmission line with feedback. Without feedback it is characterized by the "telegrapher's" equation

$$\begin{bmatrix} \dfrac{\partial e}{\partial t} \\ \\ \dfrac{\partial i}{\partial t} \end{bmatrix} = \begin{pmatrix} 0 & -\dfrac{1}{C}\dfrac{\partial}{\partial x} \\ \\ -\dfrac{1}{L}\dfrac{\partial}{\partial x} & 0 \end{pmatrix} \begin{bmatrix} e \\ \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \\ -\dfrac{J}{C} \end{bmatrix}. \tag{4.1}$$

Here $e(x, t)$ and $i(x, t)$ are the voltage and current on the line as a function of time and space, L and C represent the inductance and capacitance per unit length of the line, and $J(x, t)$ is the current density artificially induced on the line from external sources. e, i and J are assumed to take their values in $H = L_2[0, M]$ where M denotes the length of the line. These classical equations are well defined and their solution is characterized by a resolvent.

Now, consider the effect of injecting current into the line as a function of the voltage at the end of the line via

$$J_f(x, t) = g(x)e(M, t) \tag{4.2}$$

i.e. a current proportional to the voltage at the end of the line with density $g(x)$. In practice the current would be injected at a finite set of points and, hence, $g(x)$ would approximate a sum of delta functions concentrated at these points. As such, we may assume that there exists an $m < M$ such that $g(x)$

$= 0$, $x > m$. With the addition of the feedback term, equation (4.1) becomes

$$\begin{bmatrix} \dfrac{\partial e}{\partial t} \\ \\ \dfrac{\partial i}{\partial t} \end{bmatrix} = \left[\begin{pmatrix} 0 & -\dfrac{1}{C}\dfrac{\partial}{\partial x} \\ \\ -\dfrac{1}{L}\dfrac{\partial}{\partial x} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \\ -\dfrac{g(x)}{C} V_M & 0 \end{pmatrix} \right] \begin{bmatrix} e \\ \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \\ -\dfrac{J}{C} \end{bmatrix} \tag{4.3}$$

where V_M denotes the valuation functional, $V_M e = e(M, t)$. To apply the Theorem to verify the existence of a solution to (4.3) it suffices to show that the perturbing operator, $-(g(x)/C)V_M$, is strictly triangular with respect to an appropriate chain of orthoprojectors on $L_2[0, M]$.

For this purpose let

$$\mathcal{S} = \{S = (a, M], \ 0 \leq a < M\} \tag{4.4}$$

and let $\beta = \{P_S, S \text{ in } \mathcal{S}\}$ where P_S denotes multiplication by the characteristic function of S. Now

$$\chi_{(a,M]}(x)\left(-\frac{g(x)}{C} V_M e\right) = \chi_{(a,M]}(x)\left(-\frac{g(x)}{C} e(M, t)\right) \tag{4.5}$$

while

$$\begin{aligned} \chi_{(a,M]}(x)\left(-\frac{g(x)}{C} V_M \chi_{(a,M]} e\right) &= \chi_{(a,M]}(x)\left(-\frac{g(x)}{C} \chi_{(a,M]}(M) e(M, t)\right) \\ &= \chi_{(a,M]}(x)\left(-\frac{g(x)}{C} e(M, t)\right) \end{aligned} \tag{4.6}$$

verifying that $P_S B = P_S B P_S$; i.e. that the perturbation is triangular.

For partition of β, say defined by the partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = M \tag{4.7}$$

of $[0, M]$,

$$V_M \chi_{(t_{i-1}, t_i]} e = \begin{cases} 0, & i \leq n - 1, \\ e(M, t), & i = n, \end{cases} \tag{4.8}$$

hence

$$\sum_{i=1}^{n} \chi_{(t_{i-1}, t_i]}(x) g(x) V_M \chi_{(t_{i-1}, t_i]} e = \chi_{(t_{n-1}, t_n]}(x) g(x) e(M, t) \tag{4.9}$$

which is zero if $n - 1 > m$ since $g(x) = 0$ for $x > m$. As such, the partial sum is zero for all partitions which refine $(0, m, M)$ verifying that $\int dP\, B\, dP = 0$ and hence that our perturbation is strictly triangular.

Note that the above argument remains valid if we were to feedback a signal proportional to a derivative with respect to x of the voltage at the end of the line, say the 43rd.

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