PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 1, August 1975

# ABSOLUTELY CLOSED MAPS

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ABSTRACT. An example is given of a continuous function $f: X \to Y$ which is closed, has point inverses $H$-closed, but which can be extended to a continuous function $F: Z \to Y$ for some $Z$ which has $X$ as a proper dense subset. A partial characterization of nonextendable functions is given in terms similar to Bourbaki's theorem that perfect maps $f: X \to Y$ are those for which $f \times i_Z: X \times Z \to Y \times Z$ is a closed map for all spaces $Z$.

A map $f: X \longrightarrow Y$ is called *absolutely closed* if there does not exist an extension of $f$ to $F: X^* \longrightarrow Y$ where $X^*$ is any space with $X$ as a proper dense subset. These maps seem interesting in terms of extensions of functions and in their relation to $H$-closed spaces. Absolutely closed maps were first introduced by Blaszczyk and Mioduszewski [1] and characterizations have been given by Viglino [6] and Dickman [3].

All spaces are assumed Hausdorff and all maps continuous. A *countable ultrafilter* is one with a countable base. A subset $A$ of $X$ is called *(countably) far from the remainder* (c.f.f.r. or f.f.r. respectively) iff for each free open (countable) ultrafilter $\mathfrak{U}$ on $X$, there is some $U \in \mathfrak{U}$ for which $\overline{U} \cap A = \emptyset$. A closed set is *regular closed* iff it is the closure of its interior. A map is *regular closed* iff the image of every regular closed set is closed.

**Theorem 1** (Dickman [3]). *A map $f: X \longrightarrow Y$ is absolutely closed iff*
(1) *$f^{-1}(y)$ is f.f.r. for all $y \in Y$ and*
(2) *$f$ is regular closed.*

Viglino [6] has asked whether every closed map with point inverses $H$-closed is absolutely closed. In view of Theorem 1, Dickman [3] pointed out that this question may be rephrased: if point inverses of a closed map are $H$-closed, are they also f.f.r.? The following example gives a negative answer to this question. The space used is essentially the one described in [1, p. 48].

**Example.** Let $X = ([-1, 1] \times N^+) \cup \{p\}$, where $N^+$ is the set of positive integers and $p$ is an additional point whose neighborhoods are of the form

Received by the editors May 9, 1974.
*AMS (MOS) subject classifications* (1970). Primary 54C10, 54D25.
*Key words and phrases.* $H$-closed, absolutely closed map, regular closed map.

$V_n = \{(x, i) | x > 0, i \geq n\} \cup \{p\}$. Consider the subset $A$ of $X$ consisting of $([0, 1] \times N^+) \cup \{p\}$. $X$ is clearly Hausdorff and $A$ is $H$-closed but $A$ is not f.f.r. Construct an ultrafilter $\mathfrak{V}$ on $X$ as follows: let $\mathfrak{U} = \mathfrak{U}_1 \cup \mathfrak{U}_2$ where the sets in $\mathfrak{U}_1$ are $U_n = \{(x, i) | x < 0, i \geq n\}$ and the sets in $\mathfrak{U}_2$ are of the form $U_{t_1 t_2 \ldots} = \{(x, n) | t_n < x < 0, -1 \leq t_n < 0\}$. Then $\mathfrak{U}$ is contained in some free ultrafilter $\mathfrak{V}$. We claim that $\overline{V} \cap A \neq \emptyset$ for all $V \in \mathfrak{V}$. If there were some $V_0 \in \mathfrak{V}$ for which $\overline{V}_0 \cap A = \emptyset$, $\overline{V}_0 \cap \{(0, n) | n \in N^+\} = \emptyset$. It follows that for each $n$ there exist $t_n$ and $s_n$ with $-1 \leq t_n < 0$, $0 < s_n \leq 1$ and $V_0$ disjoint from the interval $(t_n, s_n)$. But then $V_0 \cap U_{t_1 t_2 \ldots} = \emptyset$, which is impossible. Thus $\overline{V} \cap A \neq \emptyset$ for all $V \in \mathfrak{V}$ and $A$ is not f.f.r.

Identify $A$ to a point $y_0$ and let $f$ be the natural map and $Y$ the quotient space. Then $Y$ is Hausdorff, $f$ is closed, $f^{-1}(y)$ is $H$-closed for all $y \in Y$ but $f^{-1}(y_0) = A$ is not f.f.r. so that $f$ is not absolutely closed.

**Remarks.** It is well known [2] that a map $f: X \to Y$ is perfect iff $f \times i_Z: X \times Z \to Y \times Z$ is a closed map for all spaces $Z$, where $i_Z$ is the identity map on $Z$. In the special case that $Y$ is a point we obtain as a corollary Scarborough's result [4] that a space $X$ is compact iff the projection $\pi_Z: X \times Z \to Z$ is closed for all spaces $Z$. Our interest here is to examine to what extent similar results hold for absolutely closed functions. That is, what theorem gives as a corollary Scarborough's result [4] that a space $X$ is $H$-closed iff the projection $\pi_Z: X \times Z \to Z$ is regular closed for all spaces $Z$? We conjecture that $f: X \to Y$ is absolutely closed iff $f \times i_Z: X \times Z \to Y \times Z$ is regular closed for all spaces $Z$. The necessity is proved in Theorem 3 but for the sufficiency we only have the partial result of Theorem 2. Notice that Theorem 2 implies that for the function of the Example, there is a space $Z$ with $f \times i_Z: X \times Z \to Y \times Z$ not regular closed.

**Definition.** A map $f: X \to Y$ is *countably absolutely closed* if there does not exist an extension of $f$ to $F: X^* \to Y$ where $X^*$ is any space with $X$ as a proper dense subset such that all points of $X^* - X$ have a countable neighborhood base.

The proofs of the following lemmas follow the techniques of Theorem 1 above and Theorem 1.2 of [6] and are omitted. Lemma 1 also holds for arbitrary countable filter-bases.

**Lemma 1.** *$f: X \to Y$ is countably absolutely closed iff no free countable ultrafilter $\mathfrak{U}$ on $X$ has $f(\mathfrak{U})$ convergent.*

**Lemma 2.** *If $f: X \to Y$ and*

(1) *$f$ is regular closed and*

(2) *$f^{-1}(y)$ is c.f.f.r. for all $y \in Y$,*
*then $f$ is countably absolutely closed. If $f$ is countably absolutely closed and $Y$ is first countable then* (1) *and* (2) *hold.*

**Theorem 2.** *Let $f: X \to Y$. If $f \times i_Z: X \times Z \to Y \times Z$ is regular closed for all spaces $Z$, then $f$ is countably absolutely closed.*

**Proof.** Since $f \times i_Z$ is regular closed for all spaces $Z$, $f$ is regular closed. Hence if $f$ is not countably absolutely closed then by Lemma 2 there is some $y_0 \in Y$ for which $f^{-1}(y_0)$ is not c.f.f.r. So, let $\mathfrak{U}$ be a free maximal countable open filter with $f^{-1}(y_0) \cap \overline{U} \neq \emptyset$ for all $U \in \mathfrak{U}$. If $\{U_i\}$ is a base for $\mathfrak{U}$ we may assume that $U_{n+1} \subseteq U_n$ for all $n$. Let $p$ be an additional point with basic neighborhoods $U_i \cup \{p\}$. Then if $Z = X \cup \{p\}$, $Z - X$ is first countable and $Z$ is Hausdorff, so consider $f \times i_Z: X \times Z \to Y \times Z$. Since $\{p\} = \bigcap \mathrm{Cl}_Z\, U_n$ we may assume $U_{n-1} - \mathrm{Cl}_Z\, U_n \neq \emptyset$ for all $n$. Let

$$V_1 = U_1 \times (Z - \mathrm{Cl}_Z U_1), \qquad V_2 = U_2 \times (U_1 - \mathrm{Cl}_Z U_2), \cdots,$$
$$V_n = U_n \times (U_{n-1} - \mathrm{Cl}_Z U_n)$$

and let $W = \overline{\bigcup_{i=1}^{\infty} V_i}$. Since $f \times i_Z$ is regular closed, $(f \times i_Z)(W)$ is closed.

We claim that if $z \neq p$, $(y_0, z) \in (f \times i_Z)(W)$. Since $z \neq p$, there is a first $k$ for which $z \notin \overline{U}_k$, hence $z \in \overline{U_{k-1} - \overline{U}_k} \subseteq \overline{U_{k-1} - \overline{U}_k}$. Now, $f^{-1}(y_0) \cap \overline{U}_j \neq \emptyset$ for all $j$, and $f$ is regular closed so that $y_0 \in f(\overline{U}_j) = \overline{f(U_j)}$ for all $j$. Therefore,

$$(y_0, z) \in \overline{f(U_k)} \times \overline{U_{k-1} - \overline{U}_k} = \overline{f(U_k) \times (U_{k-1} - \overline{U}_k)}$$
$$= \overline{(f \times i_Z)(V_k)} \subseteq \overline{(f \times i_Z)\left(\bigcup V_i\right)} = (f \times i_Z)(W),$$

since $f \times i_Z$ is regular closed. Thus, $\{y_0\} \times X \subseteq (f \times i_Z)(W)$ so that $\{y_0\} \times Z \subseteq (f \times i_Z)(W)$. It follows that there is some $x_0 \in f^{-1}(y_0)$ with $(x_0, p) \in W$.

Let $K$ be the first integer with $x_0 \notin \overline{U_K}$ and consider the neighborhood $(X - \overline{U}_K) \times U_{K+1}$ of $(x_0, p)$. If $((X - \overline{U}_K) \times U_{K+1}) \cap (U_{N+1} \times (U_N - \overline{U}_{N+1}))$ $\neq \emptyset$, then if $N \leq K$ then $U_K \subseteq U_N$ so that $U_{K+1} \cap (U_N - \overline{U_{N+1}}) = \emptyset$, while if $K < N$, $U_N \subseteq U_K$ so $U_{N+1} \cap (X - \overline{U}_K) = \emptyset$. In either case the contradiction implies that $((X - \overline{U}_k) \times U_{k+1}) \cap \bigcup V_i = \emptyset$ and hence $(x_0, p) \notin \overline{\bigcup V_i} = W$. This contradiction establishes the theorem.

In the proof of Theorem 2, if $X$ is first countable so is $Z$. Also, since a first countable subset which is c.f.f.r. is easily first-countable-and-Hausdorff-closed (closed in every first countable Hausdorff space in which it is embedded [5]) we have from Theorem 2,

**Corollary 2.1.** *If* $f \times i_Z: X \times Z \to Y \times Z$ *is regular closed for all first countable spaces and* $X$ *is first countable, then* $f^{-1}(y)$ *is first-countable-and-Hausdorff-closed for all* $y \in Y$ *and* $f$ *is countably absolutely closed.*

**Lemma 3** (Stephenson [5, Theorem 5.7]). *For each first countable Hausdorff space* $X$, *the space* $Z$ *of all countable open ultrafilters on* $X$ *is first-countable-and-Hausdorff-closed and contains* $X$ *as a dense subset.*

Notice that in this extension we can take the neighborhoods of a point $p \in Z$ to be $\{p\} \cup U_i$, where $U_i$ is a member of the base for $p$. Then, in the proof of Theorem 2 we could have taken $Z$ to be this first countable Hausdorff extension of $X$, with $p \in Z - X$.

**Corollary 2.2.** *If* $X$ *is first countable and* $f \times i_Z: X \times Z \to Y \times Z$ *is regular closed for all first-countable-and-Hausdorff-closed spaces* $Z$, *then* $f$ *is countably absolutely closed.*

The next lemma is proved by using Viglino's ultrafilter characterization of absolutely closed maps [6, Theorem 1.2] (or Lemma 1 for the countable case) and imitating the proof of Lemma 2 of [2, p. 101]. Theorem 3 and its corollary then follow from the lemma and Theorem 1, and the lemma and Lemma 2 respectively.

**Lemma 4.** *If* $f_\alpha: X_\alpha \to Y_\alpha$ *are (countably) absolutely closed for all* $\alpha$ *then so is* $f: \Pi X_\alpha \to \Pi Y_\alpha$, $(f(x))_\alpha = f_\alpha(x_\alpha)$.

**Theorem 3.** *If* $f: X \to Y$ *is absolutely closed then* $f \times i_Z: X \times Z \to Y \times Z$ *is regular closed for all spaces* $Z$.

**Corollary 3.1.** *If* $f: X \to Y$ *is countably absolutely closed and* $Y$ *is first countable,* $f \times i_Z: X \times Z \to Y \times Z$ *is regular closed for all first countable spaces* $Z$.

**Corollary 3.2.** *If* $X$ *is first countable the following are equivalent:*

(1) $\pi_Z: X \times Z \to Z$ *is regular closed for all first countable spaces* $Z$;

(2) $\pi_Z: X \times Z \to Z$ *is regular closed for all first-countable-Hausdorff-closed spaces* $Z$;

(3) $X$ *is first-countable-and-Hausdorff-closed.*

**Proof.** (1) $\Rightarrow$ (2) is obvious, (2) $\Rightarrow$ (3) follows from Corollary 2.2 by taking $Y$ to be a point, and (3) $\Longrightarrow$ (1), follows from Corollary 3.1 again by letting $Y$ be a single point.

**Added in proof.** The following is proved by using Theorem 1 and applying the methods of the proof of Theorem 7 of [4].

**Theorem.** *If $f \times i_Z\colon X \times Z \to Y \times Z$ is regular closed for all spaces $Z$, then $f$ is absolutely closed.*

**Question.** If $f \times i_Z\colon X \times Z \to Y \times Z$ is regular closed for all $H$-closed spaces $Z$, is $f$ necessarily absolutely closed?

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